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Asymptotic Solutions and Eigenvalues of a Class of Discontinuous Dirac Operators

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Abstract

In this paper, we study the asymptotics of the solutions and eigenvalues of Dirac operators defined on $[0,\pi]$ with jump conditions at point $a\in(0,\pi)$. The asymptotics of the solutions and eigenvalues were obtained, and the results to the case of n jump points were extended.

Keywords

Dirac Operator, Solution, Eigenvalue, Asymptotics

一类非连续Dirac算子解和特征值的渐近式

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摘 要

本文研究定义在$[0,\pi]$区间上，在点$a\in(0,\pi)$具有跳跃条件的Dirac算子解和特征值的渐近性，给出了解和特征值的渐近式，并将所得结论拓展到n个跳跃点的情形。

关键词

Dirac算子，解，特征值，渐近式

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1. 预备知识

考虑定义在$[0,\pi]$上的 Dirac 微分方程：

$$\hat{L}(Y)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{\mathrm{d}Y}{\mathrm{d}x}+\begin{pmatrix}-p(x) & 0\\ 0 & -r(x)\end{pmatrix}Y=\lambda Y, \tag{1}$$

满足边界条件

$$T_1(Y)=y_1(0)\sin\alpha+y_2(0)\cos\alpha=0, \tag{2}$$

$$T_2(Y)=y_1(\pi)\sin\beta+y_2(\pi)\cos\beta=0 \tag{3}$$

和跳跃条件

$$\begin{cases} y_1(a+0)=y_1(a-0), \\ y_2(a+0)=y_2(a-0)+by_1(a-0), \end{cases} \tag{4}$$

其中$Y(x)=(y_1(x),y_2(x))^{\mathrm{T}}$，$p(x),r(x)\in L_2[0,1]$为实值连续函数，$\alpha,\beta\in[0,2\pi]$，$a\in(0,\pi)$，$b\in\mathbb{R}$。

Dirac 算子又称 AKNS 算子，具有跳跃(或不连续，传递)条件的非连续 Dirac 边值问题在数学物理、力学、电子学、地球物理学和其他自然科学的分支中具有广泛应用，如地壳底部横波的反射可能使描述原系统的物理方程中的特征函数具有跳跃点。在 Dirac 算子谱理论的研究中，解和特征值的渐近估计是基本问题。文献[1]利用留数方法给出了 Dirac 算子的谱分解；文献[2]根据积分方程的方法讨论了 Dirac 算子特征值的渐近式；文献[3]借助 Prüfer 变换给出了 Dirac 问题的特征值渐近估计式。文献[4]利用摄动的方法研究了非连续 Sturm-Liouville 算子特征值的渐近式。文献[5] [6] [7] [8]研究了 Dirac 算子特征值的性质。

本文将利用 Dirac 方程解的基本性质计算问题(1)~(4)解的渐近式，借助文献[4]中摄动的方法，利用$q(x)\equiv 0$时具有跳跃条件的 Dirac 算子的特征值$\{\lambda_n^0\}_{n=0}^{+\infty}$刻画问题(1)~(4)的特征值。并将所得结论推广至原系统具有 n 个跳跃点的情形。

2. 主要结论

令$U(x)=(u_1(x),u_2(x))^{\mathrm{T}}$和$V(x)=(v_1(x),v_2(x))^{\mathrm{T}}$的分量$u_j(x),v_j(x)(j=1,2)$分别是$[0,a]$和$[a,\pi]$上的连续可微函数。记$\langle U,V\rangle=u_1v_2-u_2v_1$。若$U(x)$和$V(x)$满足跳跃条件(4)，则

$$\langle U,V\rangle\big|_{x=a+0}=\langle U,V\rangle\big|_{x=a-0}, \tag{5}$$

即函数$\langle U,V\rangle$在$[0,\pi]$上连续。若$U(x,\lambda)$和$V(x,\lambda)$分别为方程$\hat{L}U=\lambda U$和$\hat{L}V=\mu V$的解，则

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle U,V\rangle=(\lambda-\mu)\{u_1(x)v_1(x)+u_2(x)v_2(x)\}. \tag{6}$$

令$\varphi(x,\lambda)=(\varphi_1(x,\lambda),\varphi_2(x,\lambda))^{\mathrm{T}}$，$\psi(x,\lambda)=(\psi_1(x,\lambda),\psi_2(x,\lambda))^{\mathrm{T}}$，$N(x,\lambda)=(n_1(x,\lambda),n_2(x,\lambda))^{\mathrm{T}}$，$N(x,\lambda)=(n_1(x,\lambda),n_2(x,\lambda))^{\mathrm{T}}$是方程(1)的解，满足跳跃条件(4)和如下初始条件

$$\varphi_1(0,\lambda)=\cos\alpha,\varphi_2(0,\lambda)=-\sin\alpha, \tag{7}$$

$$\psi_1(\pi,\lambda)=\cos\beta,\psi_2(\pi,\lambda)=-\sin\beta, \tag{8}$$

$$m_1(0,\lambda)=1,m_2(0,\lambda)=0, \tag{9}$$

$$n_1(0,\lambda)=0,n_2(0,\lambda)=1, \tag{10}$$

则 $T_1(\varphi)=T_2(\psi)=0$。

令 $M_0(x,\lambda)=(\tilde{m}_1(x,\lambda),\tilde{m}_2(x,\lambda))^{\mathrm{T}}$ 和 $N_0(x,\lambda)=(\tilde{n}_1(x,\lambda),\tilde{n}_2(x,\lambda))^{\mathrm{T}}$ 是方程(1)的解，满足初始条件：

$$\tilde{m}_1(0,\lambda)=\tilde{n}_2(0,\lambda)=1,\quad \tilde{m}_2(0,\lambda)=\tilde{n}_1(0,\lambda)=0. \tag{11}$$

但不满足跳跃条件。令

$$\Delta(\lambda)=\langle\varphi(x,\lambda),\psi(x,\lambda)\rangle. \tag{12}$$

由文献[2]可知，$\Delta(\lambda)$ 的取值与 x 无关，函数 $\Delta(\lambda)$ 为算子 $\hat{L}$ 的特征值函数。则

$$\Delta(\lambda)=-T_1(\psi)=T_2(\varphi), \tag{13}$$

其零点为算子 $\hat{L}$ 的特征值，记为 $\sigma(\hat{L})$，$\sigma(\hat{L})=\{\lambda_n\}_{n=-\infty}^{+\infty}$。

引理 1 [1] (1) $\varphi(x,\lambda_n),\psi(x,\lambda_n)$ 为算子 $\hat{L}$ 的特征函数，且

$$\psi(x,\lambda_n)=C\varphi(x,\lambda_n),\ C\neq 0. \tag{14}$$

(2) 记

$$\alpha_n=\int_0^\pi\{\varphi^2(x,\lambda_n)+\psi^2(x,\lambda_n)\}\mathrm{d}x \tag{15}$$

为特征值 λ_n 对应的规范常数，且

$$C\alpha_n=\Delta_1(\lambda_n), \tag{16}$$

其中 $\Delta_1(\lambda_n)=\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\Delta(\lambda)$。

(3) 特征值 λ_n 是实数，特征函数 $\varphi(x,\lambda_n)$ 和 $\psi(x,\lambda_n)$ 为实值函数，$\Delta(\lambda)$ 的零点是简单的，即 $\Delta_1(\lambda_n)\neq 0$。在 L_2 空间中，不同特征值对应的特征函数是正交的。

根据跳跃条件(4)可得当 $x<a$ 时，有

$$M(x,\lambda)=M_0(x,\lambda),\ N(x,\lambda)=N_0(x,\lambda), \tag{17}$$

当 $x>a$ 时，有

$$\begin{gathered}M(x,\lambda)=A_1M_0(x,\lambda)+B_1N_0(x,\lambda),\\ N(x,\lambda)=C_1M_0(x,\lambda)+D_1N_0(x,\lambda),\end{gathered} \tag{18}$$

其中

$$\begin{cases}A_1=\tilde{m}_1(a,\lambda)\tilde{n}_2(a,\lambda)-\tilde{m}_2(a,\lambda)\tilde{n}_1(a,\lambda)-b\tilde{m}_1(a,\lambda)\tilde{n}_1(a,\lambda),\\ B_1=b\tilde{m}_1^2(a,\lambda),\\ C_1=-b\tilde{n}_1^2(a,\lambda),\\ D_1=\tilde{m}_1(a,\lambda)\tilde{n}_2(a,\lambda)-\tilde{m}_2(a,\lambda)\tilde{n}_1(a,\lambda)+b\tilde{m}_1(a,\lambda)\tilde{n}_1(a,\lambda).\end{cases} \tag{19}$$

由文献[2]可知，函数 $M_0(x,\lambda)=\left(\tilde{m}_1(x,\lambda),\tilde{m}_2(x,\lambda)\right)^{\mathrm{T}}$ 和 $N_0(x,\lambda)=\left(\tilde{n}_1(x,\lambda),\tilde{n}_2(x,\lambda)\right)^{\mathrm{T}}$ 满足下列积分方程：

$$\tilde{m}_1(x,\lambda)=\cos\lambda x-\int_0^x\tilde{m}_1(\tau,\lambda)p(\tau)\sin\lambda(x-\tau)\mathrm{d}\tau-\int_0^x\tilde{m}_2(\tau,\lambda)r(\tau)\cos\lambda(x-\tau)\mathrm{d}\tau, \tag{20}$$

$$\tilde{m}_2(x,\lambda)=\sin\lambda x-\int_0^x\tilde{m}_2(\tau,\lambda)r(\tau)\sin\lambda(x-\tau)\mathrm{d}\tau+\int_0^x\tilde{m}_1(\tau,\lambda)p(\tau)\cos\lambda(x-\tau)\mathrm{d}\tau, \tag{21}$$

$$\tilde{n}_1(x,\lambda)=-\sin\lambda x-\int_0^x\tilde{n}_1(\tau,\lambda)p(\tau)\sin\lambda(x-\tau)\mathrm{d}\tau-\int_0^x\tilde{n}_2(\tau,\lambda)r(\tau)\cos\lambda(x-\tau)\mathrm{d}\tau, \tag{22}$$

$$\tilde{n}_2(x,\lambda)=\cos\lambda x-\int_0^x\tilde{n}_2(\tau,\lambda)r(\tau)\sin\lambda(x-\tau)\mathrm{d}\tau+\int_0^x\tilde{n}_1(\tau,\lambda)p(\tau)\cos\lambda(x-\tau)\mathrm{d}\tau. \tag{23}$$

当 $|\lambda|\to\infty$ 时，$M_0(x,\lambda)$ 和 $N_0(x,\lambda)$ 的渐近式如下：

$$M_0(x,\lambda)=\begin{pmatrix}\cos\left\{\lambda x-\frac{1}{2}\int_0^x\left[p(\tau)+r(\tau)\right]\mathrm{d}\tau\right\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right)\\ \sin\left\{\lambda x-\frac{1}{2}\int_0^x\left[p(\tau)+r(\tau)\right]\mathrm{d}\tau\right\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right)\end{pmatrix}, \tag{24}$$

$$N_0(x,\lambda)=\begin{pmatrix}-\sin\left\{\lambda x-\frac{1}{2}\int_0^x\left[p(\tau)+r(\tau)\right]\mathrm{d}\tau\right\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right)\\ \cos\left\{\lambda x-\frac{1}{2}\int_0^x\left[p(\tau)+r(\tau)\right]\mathrm{d}\tau\right\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right)\end{pmatrix}, \tag{25}$$

其中 $\lambda=\sigma+i\tau$，且 $\sigma,\tau\in\mathbb{R}$。由(20)~(23)可得

$$\begin{aligned}\tilde{m}_1(x,\lambda)=&\cos\left\{\lambda x-\xi(x)\right\}-\frac{1}{4\lambda}\left\{p(0)-r(0)\right\}\cos\left\{\lambda x-\xi(x)\right\}\\&+\frac{1}{4\lambda}\left\{p(x)-r(x)\right\}\cos\left\{\lambda x-\xi(x)\right\}\\&+\frac{1}{8\lambda}\int_0^x\left\{p(\tau)-r(\tau)\right\}^2\mathrm{d}\tau\sin\left\{\lambda x-\xi(x)\right\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{\lambda^2}\right),\end{aligned} \tag{26}$$

$$\begin{aligned}\tilde{m}_2(x,\lambda)=&\sin\left\{\lambda x-\xi(x)\right\}-\frac{1}{4\lambda}\left\{p(x)-r(x)\right\}\sin\left\{\lambda x-\xi(x)\right\}\\&-\frac{1}{4\lambda}\left\{p(0)-r(0)\right\}\sin\left\{\lambda x-\xi(x)\right\}\\&-\frac{1}{8\lambda}\int_0^x\left\{p(\tau)-r(\tau)\right\}^2\mathrm{d}\tau\cos\left\{\lambda x-\xi(x)\right\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{\lambda^2}\right),\end{aligned} \tag{27}$$

$$\begin{aligned}\tilde{n}_1(x,\lambda)=&-\sin\left\{\lambda x-\xi(x)\right\}-\frac{1}{4\lambda}\left\{p(0)-r(0)\right\}\sin\left\{\lambda x-\xi(x)\right\}\\&-\frac{1}{4\lambda}\left\{p(x)-r(x)\right\}\sin\left\{\lambda x-\xi(x)\right\}\\&+\frac{1}{8\lambda}\int_0^x\left\{p(\tau)-r(\tau)\right\}^2\mathrm{d}\tau\cos\left\{\lambda x-\xi(x)\right\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{\lambda^2}\right),\end{aligned} \tag{28}$$

$$\begin{aligned}\tilde{n}_2(x,\lambda)=&\cos\{\lambda x-\xi(x)\}-\frac{1}{4\lambda}\{p(x)-r(x)\}\cos\{\lambda x-\xi(x)\}\\&+\frac{1}{4\lambda}\{p(0)-r(0)\}\cos\{\lambda x-\xi(x)\}\\&+\frac{1}{8\lambda}\int_0^x\{p(\tau)-r(\tau)\}^2\mathrm{d}\tau\sin\{\lambda x-\xi(x)\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{\lambda^2}\right).\end{aligned}\tag{29}$$

其中 $\xi(x)=\frac{1}{2}\int_0^x[p(\tau)+r(\tau)]\mathrm{d}\tau$ 。由(19)和(26)~(29)可得

$$\begin{cases}A_1=1+\frac{b}{2}\sin 2\{\lambda a-\xi(a)\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right),\\B_1=\frac{b}{2}\left(\cos 2\{\lambda a-\xi(a)\}+1\right)+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right),\\C_1=\frac{b}{2}\left(\cos 2\{\lambda a-\xi(a)\}-1\right)+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right),\\D_1=1-\frac{b}{2}\sin 2\{\lambda a-\xi(a)\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right).\end{cases}\tag{30}$$

因为

$$\varphi(x,\lambda)=\cos\alpha M(x,\lambda)-\sin\alpha N(x,\lambda),\tag{31}$$

故通过(17)~(18)和(26)~(30)计算可得

$$\varphi_1(x,\lambda)=\begin{cases}\cos\{\lambda x-\xi(x)-\alpha\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right), & x\in(0,a),\\\cos\{\lambda x-\xi(x)-\alpha\}-\frac{b}{2}\sin\{\lambda x-\xi(x)-\alpha\}\\\quad-\frac{b}{2}\sin\{\lambda x-\xi(x)-2[\lambda a-\xi(a)]+\alpha\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right), & x\in(a,\pi),\end{cases}\tag{32}$$

$$\varphi_2(x,\lambda)=\begin{cases}\sin\{\lambda x-\xi(x)-\alpha\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right), & x\in(0,a),\\\sin\{\lambda x-\xi(x)-\alpha\}+\frac{b}{2}\cos\{\lambda x-\xi(x)-\alpha\}\\\quad+\frac{b}{2}\cos\{\lambda x-\xi(x)-2[\lambda a-\xi(a)]+\alpha\}+O\left(\frac{\mathrm{e}^{|\tau|x}}{|\lambda|}\right), & x\in(a,\pi).\end{cases}\tag{33}$$

Dirac 算子(1)~(4)的特征值函数为

$$\Delta(\lambda)=\varphi_1(\pi,\lambda)\sin\beta+\varphi_2(\pi,\lambda)\cos\beta\tag{34}$$

由(32)~(34)可得

$$\begin{aligned}\Delta(\lambda)=&\sin\{\lambda\pi-\xi(\pi)-\alpha+\beta\}+\frac{b}{2}\cos\{\lambda\pi-\xi(\pi)-\alpha+\beta\}\\&+\frac{b}{2}\cos\{\lambda\pi-\xi(\pi)-2[\lambda a-\xi(a)]+\alpha+\beta\}+O\left(\mathrm{e}^{|\tau|x}\right).\end{aligned}\tag{35}$$

下面求特征值 λ_n 的渐近式。令 λ_n^0 是函数

$$\Delta^0(\lambda)=\sin\{\lambda\pi-\xi(\pi)-\alpha+\beta\}. \tag{36}$$

的零点。则

$$\lambda_n^0=n+\frac{\vartheta}{\pi}+O\left(\frac{1}{n}\right), \tag{37}$$

此处 $\vartheta=\alpha-\beta+\frac{1}{2}\int_0^\pi\left[p(\tau)+r(\tau)\right]\mathrm{d}\tau$。Dirac 算子(1)~(4)的特征值具有如下的渐近形式：

$$\lambda_n=\lambda_n^0+o(1),\quad n\to\infty. \tag{38}$$

将(32)和(33)代入规范常数(15)可得

$$\alpha_n=\alpha_n^0+o(1),\quad n\to\infty, \tag{39}$$

此处 $\alpha_n^0=\pi+O\left(\frac{1}{n}\right)$，即 $\alpha_n=O(1),(\alpha_n)^{-1}=O(1)$。由(16)可得

$$\left|\Delta_1(\lambda_n)\right|\asymp\tilde{C}. \tag{40}$$

即 $\left|\Delta_1(\lambda_n)\right|$ 等价于常数 $\tilde{C}$。

下面计算特征值更为精确的渐近式。令(35)式 $\Delta(\lambda_n)=0$，可得

$$\sin\{\lambda_n\pi-\xi(\pi)-\alpha+\beta\}=O\left(\frac{1}{\lambda_n^0}\right). \tag{41}$$

根据(38)式可得

$$\lambda_n=\lambda_n^0+\varepsilon_n,\varepsilon_n\to 0. \tag{42}$$

因为

$$\Delta^0(\lambda_n^0)=\sin\{\lambda_n^0\pi-\xi(\pi)-\alpha+\beta\}=0, \tag{43}$$

故由(41)可得

$$\varepsilon_n\pi\cos\{\lambda_n^0\pi-\xi(\pi)-\alpha+\beta\}=O\left(\frac{1}{\lambda_n^0}\right)+O(\varepsilon_n^2). \tag{44}$$

由(36)可得

$$\Delta_1^0(\lambda_n^0):=\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta^0(\lambda)\right)\Bigg|_{\lambda=\lambda_n^0}=\pi\cos\{\lambda_n^0\pi-\xi(\pi)-\alpha+\beta\}. \tag{45}$$

因此

$$\varepsilon_n\Delta_1^0(\lambda_n^0)=O\left(\frac{1}{\lambda_n^0}\right)+O(\varepsilon_n^2). \tag{46}$$

由(40)可得 $\left|\Delta_1^0(\lambda_n^0)\right|\asymp\tilde{C}$，故

$$\varepsilon_n=O\left(\frac{1}{\lambda_n^0}\right). \tag{47}$$

将(35)代入$\Delta(\lambda_n)=0$，根据(47)可得

$$\lambda_n=\lambda_n^0+\frac{\theta_n}{\lambda_n^0}+\frac{\kappa_n}{\lambda_n^0}. \tag{48}$$

其中$\kappa_n=o(1)$，且

$$\theta_n=\left(\frac{b}{2}\cos\left\{\lambda_n^0\pi-\xi(\pi)-\alpha+\beta\right\}+\frac{b}{2}\cos\left\{\lambda_n^0\pi-\xi(\pi)-2\left[\lambda_n^0a-\xi(a)\right]+\alpha+\beta\right\}\right)\left(\Delta_1^0\left(\lambda_n^0\right)\right)^{-1}. \tag{49}$$

3. 具有 *n* 个跳跃点的 Dirac 方程

考虑问题(1)~(3)，将(4)的跳跃点增至有限多个，$a_i\in(0,1), i=1,2,3,\cdots,n(n<+\infty)$，即满足跳跃条件

$$\begin{cases}y_1(a_i+0)=y_1(a_i-0),\\ y_2(a_i+0)=y_2(a_i-0)+b_iy_1(a_i-0),\end{cases} \tag{50}$$

其中$b_i\in\mathbb{R}$。

令$\varphi(x,\lambda)=(\varphi_1(x,\lambda),\varphi_2(x,\lambda))^{\mathrm{T}}$，$\psi(x,\lambda)=(\psi_1(x,\lambda),\psi_2(x,\lambda))^{\mathrm{T}}$，$M(x,\lambda)=(m_1(x,\lambda),m_2(x,\lambda))^{\mathrm{T}}$，$N(x,\lambda)=(n_1(x,\lambda),n_2(x,\lambda))^{\mathrm{T}}$是方程(1)满足初始条件

$$\varphi_1(0,\lambda)=\cos\alpha,\varphi_2(0,\lambda)=-\sin\alpha, \tag{51}$$

$$\psi_1(\pi,\lambda)=\cos\beta,\psi_2(\pi,\lambda)=-\sin\beta, \tag{52}$$

$$m_1(0,\lambda)=1,m_2(0,\lambda)=0, \tag{53}$$

$$n_1(0,\lambda)=0,n_2(0,\lambda)=1, \tag{54}$$

和跳跃条件(50)的解。

令$M_0(x,\lambda)=(\tilde{m}_1(x,\lambda),\tilde{m}_2(x,\lambda))^{\mathrm{T}}$和$N_0(x,\lambda)=(\tilde{n}_1(x,\lambda),\tilde{n}_2(x,\lambda))^{\mathrm{T}}$是方程(1)的解，即不满足跳跃条件，满足如下初始条件：

$$\tilde{m}_1(0,\lambda)=\tilde{n}_2(0,\lambda)=1,\tilde{m}_2(0,\lambda)=\tilde{n}_1(0,\lambda)=0. \tag{55}$$

根据跳跃条件(50)可得

$$M(x,\lambda)=M_0(x,\lambda),N(x,\lambda)=N_0(x,\lambda),\quad x\in(0,a_1), \tag{56}$$

$$\begin{aligned}M(x,\lambda)&=M(x,\lambda,a_1)=A_1M_0(x,\lambda)+B_1N_0(x,\lambda),\\ N(x,\lambda)&=N(x,\lambda,a_1)=C_1M_0(x,\lambda)+D_1N_0(x,\lambda),\quad x\in(a_1,a_2),\end{aligned} \tag{57}$$

$$\begin{aligned}M(x,\lambda)&=M(x,\lambda,a_i)=A_iM(x,\lambda,a_{i-1})+B_iN(x,\lambda,a_{i-1}),\\ N(x,\lambda)&=N(x,\lambda,a_i)=C_iM(x,\lambda,a_{i-1})+D_iN(x,\lambda,a_{i-1}),\quad x\in(a_i,a_{i+1}),\end{aligned} \tag{58}$$

$$\begin{aligned}M(x,\lambda)&=M(x,\lambda,a_n)=A_nM(x,\lambda,a_{n-1})+B_nN(x,\lambda,a_{n-1}),\\ N(x,\lambda)&=N(x,\lambda,a_n)=C_nM(x,\lambda,a_{n-1})+D_nN(x,\lambda,a_{n-1}),\quad x\in(a_n,\pi).\end{aligned} \tag{59}$$

在(58)和(59)中，$i=2,3,\cdots,n-1$，且(59)式可表示如下：

$$\begin{pmatrix}M(x,\lambda,a_n)\\ N(x,\lambda,a_n)\end{pmatrix}=\begin{pmatrix}A_n & B_n\\ C_n & D_n\end{pmatrix}\begin{pmatrix}A_{n-1} & B_{n-1}\\ C_{n-1} & D_{n-1}\end{pmatrix}\cdots\begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}\begin{pmatrix}M_0(x,\lambda)\\ N_0(x,\lambda)\end{pmatrix}. \tag{60}$$

定义函数 $A(i),B(i),C(i),D(i)\ (i=1,2,3,\cdots,n)$ 为

$$\begin{cases} A(i)=\tilde{m}_1(a_i,\lambda)\tilde{n}_2(a_i,\lambda)-\tilde{m}_2(a_i,\lambda)\tilde{n}_1(a_i,\lambda)-b_i\tilde{m}_1(a_i,\lambda)\tilde{n}_1(a_i,\lambda),\\ B(i)=b_i\tilde{m}_1^2(a_i,\lambda),\\ C(i)=-b_i\tilde{n}_1^2(a_i,\lambda),\\ D(i)=\tilde{m}_1(a_i,\lambda)\tilde{n}_2(a_i,\lambda)-\tilde{m}_2(a_i,\lambda)\tilde{n}_1(a_i,\lambda)+b_i\tilde{m}_1(a_i,\lambda)\tilde{n}_1(a_i,\lambda). \end{cases} \tag{61}$$

由跳跃条件(50)计算可知，$A_1=A(1),B_1=B(1),C_1=C(1),D_1=D(1)$，当 $i=2,3,\cdots,n$ 时：

$$\begin{pmatrix} A_i\\ B_i\\ C_i\\ D_i \end{pmatrix}=\begin{pmatrix} A_{i-1} & C_{i-1} & 0 & 0\\ B_{i-1} & D_{i-1} & 0 & 0\\ 0 & 0 & A_{i-1} & C_{i-1}\\ 0 & 0 & B_{i-1} & D_{i-1} \end{pmatrix}^{-1}\begin{pmatrix} A(i)\\ B(i)\\ C(i)\\ D(i) \end{pmatrix}. \tag{62}$$

当 $i=1,2,3,\cdots,n$ 时，则有

$$\begin{cases} M(x,\lambda,a_i)=A(i)M_0(x,\lambda)+B(i)N_0(x,\lambda),\\ N(x,\lambda,a_i)=C(i)M_0(x,\lambda)+D(i)N_0(x,\lambda). \end{cases} \tag{63}$$

由(26)~(29)和(61)可得

$$\begin{cases} A(i)=1+\dfrac{b_i}{2}\sin 2\{\lambda a_i-\xi(a_i)\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right),\\ B(i)=\dfrac{b_i}{2}\left(\cos 2\{\lambda a_i-\xi(a_i)\}+1\right)+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right),\\ C(i)=\dfrac{b_i}{2}\left(\cos 2\{\lambda a_i-\xi(a_i)\}-1\right)+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right),\\ D(i)=1-\dfrac{b_i}{2}\sin 2\{\lambda a_i-\xi(a_i)\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right). \end{cases} \tag{64}$$

因为

$$\varphi(x,\lambda)=\cos\alpha M(x,\lambda)-\sin\alpha N(x,\lambda), \tag{65}$$

故由(56)~(61)和(64)计算可得

$$\varphi_1(x,\lambda)=\begin{cases} \cos\{\lambda x-\xi(x)-\alpha\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right), & x\in(0,a_1),\\ \cos\{\lambda x-\xi(x)-\alpha\}-\dfrac{b_i}{2}\sin\{\lambda x-\xi(x)-\alpha\} & \\ \quad-\dfrac{b_i}{2}\sin\{\lambda x-\xi(x)-2[\lambda a_i-\xi(a_i)]+\alpha\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right), & x\in(a_i,a_{i+1}),\\ \cos\{\lambda x-\xi(x)-\alpha\}-\dfrac{b_n}{2}\sin\{\lambda x-\xi(x)-\alpha\} & \\ \quad-\dfrac{b_n}{2}\sin\{\lambda x-\xi(x)-2[\lambda a_n-\xi(a_n)]+\alpha\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right), & x\in(a_n,\pi), \end{cases} \tag{66}$$

$$\varphi_2(x,\lambda)=\begin{cases}\sin\{\lambda x-\xi(x)-\alpha\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right), & x\in(0,a_1),\\ \sin\{\lambda x-\xi(x)-\alpha\}+\dfrac{b_i}{2}\cos\{\lambda x-\xi(x)-\alpha\} & \\ \quad+\dfrac{b_i}{2}\cos\{\lambda x-\xi(x)-2[\lambda a_i-\xi(a_i)]+\alpha\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right), & x\in(a_i,a_{i+1}),\\ \sin\{\lambda x-\xi(x)-\alpha\}+\dfrac{b_n}{2}\cos\{\lambda x-\xi(x)-\alpha\} & \\ \quad+\dfrac{b_n}{2}\cos\{\lambda x-\xi(x)-2[\lambda a_n-\xi(a_n)]+\alpha\}+O\left(\dfrac{e^{|\tau|x}}{|\lambda|}\right), & x\in(a_n,\pi).\end{cases}\tag{67}$$

Dirac 算子(1)~(3)和(50)的特征值函数为

$$\Delta(\lambda)=\varphi_1(\pi,\lambda)\sin\beta+\varphi_2(\pi,\lambda)\cos\beta.\tag{68}$$

由(66)~(68)可得

$$\begin{aligned}\Delta(\lambda)=&\sin\{\lambda\pi-\xi(\pi)-\alpha+\beta\}+\frac{b_n}{2}\cos\{\lambda\pi-\xi(\pi)-\alpha+\beta\}\\&+\frac{b_n}{2}\cos\{\lambda\pi-\xi(\pi)-2[\lambda a_n-\xi(a_n)]+\alpha+\beta\}.\end{aligned}\tag{69}$$

具有 n 个跳跃点的 Dirac 方程特征值渐近式的求法与第二部分相似，计算可得

$$\lambda_n=\lambda_n^0+\frac{\theta_n}{\lambda_n^0}+\frac{\kappa_n}{\lambda_n^0}.\tag{70}$$

其中 $\kappa_n=o(1)$，且

$$\theta_n=\left(\frac{b_n}{2}\cos\{\lambda_n^0\pi-\xi(\pi)-\alpha+\beta\}+\frac{b_n}{2}\cos\{\lambda_n^0\pi-\xi(\pi)-2[\lambda_n^0a_n-\xi(a_n)]+\alpha+\beta\}\right)\left(\Delta_1^0(\lambda_n^0)\right)^{-1}.\tag{71}$$

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