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H-Toeplitz算子的代数性质

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摘 要

本文主要研究 Bergman 空间上 H-Toeplitz 算子的代数性质。第一章介绍了相关的研究背景、基本概念及一些主要结果。第二章给出了本文主要结果的证明，证明了拟齐次符号 H-Toeplitz 算子的复对称性。

关键词

Bergman空间，H-Toeplitz算子，复对称性

Algebraic Properties of H-Toeplitz Operator

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Abstract

In this paper, we mainly study the algebraic properties of H-toeplitz operators on Bergman Spaces. In Chapter 1, we introduce the related research background, basic

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concepts and some main results. In Chapter 2, the proof of the main results of this paper is given, and the complex symmetry of quasi homogeneous signed H-toeplitz operators is proved.

Keywords

Bergman Space, H-Toeplitz Operator, Complex Symmetry

1. 介绍

记 D 为复平面 C 内的单位开圆盘 $dA(z) = \frac{1}{\pi} r dr d\theta$ 为 D 上规范化的 Lebesgue 面积测度. 记 $H(D)$ 为 D 上所有解析函数, 记 Bergman 空间 $L^2_a(D)$ 为 D 上所有勒贝格平方可积解析函数所成的 Hilbert 空间, 其内积定义为

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)} dA(z),\ \ f, g \in L^2(D).$$

空间 $L^2_a(D)$ 是 Hilbert 空间 $L^2(D, dA)$ 的闭子空间. 对于非负整数 n, 设 $e_n(z) = \sqrt{n+1} z^n$. 对所有 $z \in D$. 集合 $\{e_n\}_{n\geq 0}$ 构成 $L^2(D)$ 的一组标准正交基 [1]. 对于 $z, w \in D$, Bergman 空间中的再生核称为 Bergman 核, 定义为

$$K_z(w) = K(z, w) = \frac{1}{(1-\bar{z}w)^2}.$$

记 $P_{L^2_a} : L^2(D, dA) \to L^2_a(D)$, 表示空间 $L^2(D, dA)$ 到空间 $L^2_a(D)$ 的正交投影, 称为 Bergman 投影. 对所有 $f \in L^2(D, dA)$, 由再生核性质:

$$P_{L^2_a}(f)(z) = \langle P_{L^2_a} f, K_z\rangle = \langle f, K_z\rangle = \int_D \frac{f(w)}{(1-z\overline{w})^2} dA(w).$$

见 [2].对 $\phi \in L^\infty(D)$, 乘法算子 M_ϕ 定义为 $M_\phi(f) = \phi f$. 符号为 $\phi \in L^\infty(D)$ 的 Toeplitz 算子 $T_\phi : L^2_a(D) \to L^2_a(D)$ 由下式定义

$$T_\phi(f) = P_{L^2_a}(\phi f) = \int_D \frac{\phi(z) f(w)}{(1-\bar{z}w)^2} dA(w), f \in L^2(D).$$

对 $\phi \in L^\infty(D)$, Hankel 算子 $H_\phi : L^2_a(D) \to L^2_a(D)$ 定义为

$$H_\phi(f) = P_{L^2_a} M_\phi J(f), \forall f \in L^2_a(D),$$

其中算子 $J : L^2_a(D) \to \overline{L^2_a(D)}$ 由 $J(e_n(z)) = \overline{e_{n+1}(z)}$ 给出, 其中 n 为非负整数. T_ϕ 和 H_ϕ 是 $L^2_a(D)$ 上的有界线性算子.

引理 1.1 *在 Bergman 空间 $L^2_a(D)$ 中对于非负整数 s 和 t, 以下成立,见 [3]:*

$$\langle z^s, z^t \rangle = \begin{cases} \frac{1}{s+1}, & s{=}t; \\ 0, & s \neq t. \end{cases}$$

$$P_{L^2_a}(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t}, & s \geq t; \\ 0, & s < t. \end{cases}$$

为了定义 $L^2_a(D)$ 上的 H-Toeplitz 算子的概念, 首先定义线性算子 K: $L^2_a(D) \to L^2_{harm}(D)$,

$$K(e_{2n}) = e_n, \quad K(e_{2n+1}) = \overline{e_{n+1}}, \quad n = 0, 1, 2, \cdots .$$

易见算子 K 在 $L^2_a(D)$ 上有界. 此外, $\|K\| = 1$. 算子 K 的伴随由下式给出: 对 $\forall n \geq 0$,

$$K^*(e_n) = e_{2n}, \quad K^*(\overline{e_{n+1}}) = e_{2n+1}.$$

对 $\phi \in L^\infty(D)$, H-Toeplitz 算子定义为 $B_\phi : L^2_a(D) \to L^2_a(D)$,

$$B_\phi(f) = P_{L^2_a} M_\phi K(f), f \in L^2_a(D),$$

对于每个非负整数 n, 我们有

$$B_\phi(e_{2n}) = P_{L^2_a} M_\phi K(e_{2n}) = P_{L^2_a} M_\phi(e_n) = T_\phi(e_n)$$

和

$$B_\phi(e_{2n+1}) = P_{L^2_a} M_\phi K(e_{2n+1}) = P_{L^2_a} M_\phi(\overline{e_{n+1}}) = P_{L^2_a} M_\phi J(e_n) = H_\phi(e_n).$$

故 H-Toeplitz 算子与 Toeplitz 算子与 Hankel 算子密切相关.见 [4] [5].

2. 主要结果及证明

关于 $L^2_a(D)$ 的标准正交基 $\{e_n\}_{n\geq 0}$, T_ϕ 矩阵的 $(m,n)^{th}$, 由下式给出:

$$
\begin{aligned}
\langle T_\phi(e_n), e_m\rangle &= \langle P_{L_a^2}(\phi e_n, e_m\rangle \\
&= \sqrt{n+1}\sqrt{m+1}\langle(\sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j) z^n, z^m\rangle \\
&= \sqrt{n+1}\sqrt{m+1}(\sum_{i=0}^{\infty} a_i \langle z^{i+n}, z^m\rangle + \sum_{j=1}^{\infty}\langle b_j z^n, z^{m+j}\rangle).
\end{aligned}
$$

分以下两种情形:

情形(1): 如果 $m \geq n$, 我们有:

$$
\begin{aligned}
\langle T_\phi(e_n), e_m\rangle &= \sqrt{n+1}\sqrt{m+1}\sum_{i=0}^{\infty} a_i \langle z^{i+n}, z^m\rangle \\
&= \sqrt{n+1}\sqrt{m+1}\frac{1}{m+1} a_{m-n} \\
&= \sqrt{\frac{n+1}{m+1}} a_{m-n}.
\end{aligned}
$$

情形(2): 如果 $m < n$, 我们有:

$$
\begin{aligned}
\langle T_\phi(e_n), e_m\rangle &= \sqrt{n+1}\sqrt{m+1}\sum_{j=1}^{\infty}\langle b_j z^n, z^{m+j}\rangle \\
&= \sqrt{n+1}\sqrt{m+1}\frac{1}{n+1} b_{n-m} \\
&= \sqrt{\frac{m+1}{n+1}} b_{n-m}.
\end{aligned}
$$

因此，我们可以得到:

$$
\langle T_\phi(e_n), e_m\rangle = \begin{cases} \sqrt{\frac{n+1}{m+1}} a_{m-n}, & m \geq n; \\ \sqrt{\frac{m+1}{n+1}} b_{n-m}, & m \leq n. \end{cases}
$$

其中 m 和 n 是非负整数, H_ϕ 的矩阵 $(m, n)^{th}$ 相对于 $L_a^2(D)$ 的标准正交基 $\{e_n\}_{n\geq 0}$ 由下式给出:

$$
\begin{aligned}
\langle H_\phi(e_n), e_m\rangle &= \langle P_{L_a^2} M_\phi J(e_n), e_m\rangle \\
&= \sqrt{n+2}\sqrt{m+1}\langle(\sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j)\overline{z}^{n+1}, z^m\rangle \\
&= \sqrt{n+2}\sqrt{m+1}(\sum_{i=0}^{\infty} a_i \langle z^i, z^{m+n+1}\rangle + \sum_{j=1}^{\infty} b_j \langle \overline{z}^j, z^{m+n+1}\rangle) \\
&= \frac{\sqrt{m+1}\sqrt{n+2}}{m+n+2} a_{m+n+1}.
\end{aligned}
$$

同理,对于非负整数 m 和 n ,有

$$\langle B_\phi(e_{2n}), e_m\rangle = \langle T_\phi(e_n), e_m\rangle \begin{cases} \sqrt{\frac{n+1}{m+1}}a_{m-n}, & m \geq n; \\ \sqrt{\frac{m+1}{n+1}}b_{n-m}, & m \leq n. \end{cases}$$

和

$$\langle B_\phi(e_{2n+1}), e_m\rangle = \langle H_\phi(e_n), e_m\rangle = \frac{\sqrt{m+1}\sqrt{n+2}}{m+n+2}a_{m+n+1},$$

其中 m 和 n 是非负整数, 因此, B_ϕ 关于$L_a^2(D)$ 的标准正交基 $\{e_n\}_{n\geq 0}$ 的矩阵表示由下式给出:

$$B_\phi = \begin{bmatrix} a_0 & \frac{1}{\sqrt{2}}a_1 & \frac{1}{\sqrt{2}}b_1 & \frac{1}{\sqrt{3}}a_2 & \frac{1}{\sqrt{3}}b_2 & \frac{1}{2}a_3 & \frac{1}{2}b_3 & \cdots \\ \frac{1}{\sqrt{2}}a_1 & \frac{1}{\sqrt{3}}a_2 & a_0 & \frac{\sqrt{6}}{4}a_3 & \sqrt{\frac{2}{3}}b_1 & \frac{2\sqrt{2}}{5}a_4 & \frac{1}{\sqrt{2}}b_2 & \cdots \\ \frac{1}{\sqrt{3}}a_2 & \frac{\sqrt{6}}{4}a_3 & \sqrt{\frac{2}{3}}a_1 & \frac{3}{5}a_4 & a_0 & \frac{1}{\sqrt{3}}a_5 & \frac{\sqrt{3}}{2}b_1 & \cdots \\ \frac{1}{2}a_3 & \frac{2\sqrt{2}}{5}a_4 & \frac{1}{\sqrt{2}}a_2 & \frac{1}{\sqrt{3}}a_5 & 3\frac{\sqrt{3}}{2}a_1 & \frac{4}{7}a_6 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

它的伴随矩阵如下:

$$B_\phi^* = \begin{bmatrix} \overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{3}}\overline{a_2} & \frac{1}{2}\overline{a_3} & \cdots \\ \frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \cdots \\ \frac{1}{\sqrt{2}}\overline{b_1} & \overline{a_0} & \frac{\sqrt{3}}{2}\overline{a_1} & \frac{1}{\sqrt{2}}\overline{a_2} & \cdots \\ ]\frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{3}{5}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \cdots \\ \frac{1}{\sqrt{3}}\overline{b_2} & \frac{\sqrt{3}}{2}\overline{b_1} & \overline{a_0} & \frac{\sqrt{3}}{2}\overline{a_1} & \cdots \\ \frac{1}{2}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \frac{4}{7}\overline{a_6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

空间 $L^2(D, dA)$ 上的复共轭算子 $C : L^2(D) \to L^2(D)$, 对于 $f, g \in L^2(D, dA)$, 满足 $C^2 = I$ 和$\langle Cf, Cg\rangle = \langle g, f\rangle$ 都成立. [6] 证明了 $L^2(D, dA)$ 存在一个正交基 $\{e_n\}_{n=0}^\infty$, 使得 $Ce_i = e_i$, $i = 0, 1, \cdots$. 若 $L^2(D, dA)$ 上的一个有界线性算子 T, 如果存在复共轭 C, 使得 $CT = T^*C$, 则称 T 是复对称的.见 [7].

定理 2.1 令 $\phi \in L^\infty(D)$ 为调和符号. 那么 *H-Toeplitz* 算子 B_ϕ 在 $L^2(D, dA)$ 上关于 C 为复对称算子当且仅当 $\phi = 0$.

证明. 设 $\phi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{j=0}^\infty b_j \overline{z}^j \in L^\infty(D)$ 为 D 上的调和函数, 并设 B_ϕ 是复对称算子. 这

意味着 $CB_\phi C = B_\phi^*$. 对于每个非负整数 n, 通过引理1.1 我们可以计算出:

$$
\begin{aligned}
\langle CB_\phi Ce_{2n}, e_m\rangle &= \langle Ce_m, B_\phi Ce_{2n}\rangle \\
&= \langle e_m, B_\phi e_{2n}\rangle = \langle e_m, P_{L_a^2} M_\phi K(e_{2n})\rangle \\
&= \langle e_m, P_{L_a^2} M_\phi (e_n)\rangle = \langle e_m, M_\phi (e_n)\rangle \\
&= \langle \sqrt{m+1} z^m, (\sum_{i=0}^{\infty} a_i z^i + \sum_{j=0}^{\infty} b_j \overline{z}^j)\sqrt{n+1} z^n\rangle \\
&= \sqrt{m+1}\sqrt{n+1}\langle z^m, \sum_{i=0}^{\infty} a_i z^i z^n\rangle + \sqrt{m+1}\sqrt{n+1}\langle z^m, \sum_{j=0}^{\infty} b_j \overline{z}^j z^n\rangle.
\end{aligned}
$$

因此, 出现了以下两种情形:

情形(1): 如果 $m \geq n$, 那么

$$
\begin{aligned}
\langle CB_\phi Ce_{2n}, e_m\rangle &= \langle \sqrt{m+1}\sqrt{n+1} z^m, \sum_{i=0}^{\infty} a_i z^i z^n\rangle \\
&= \sqrt{m+1}\sqrt{n+1}\sum_{i=0}^{\infty} \overline{a_i}\langle z^m, z^{i+n}\rangle \\
&= \sqrt{m+1}\sqrt{n+1}\frac{1}{m+1}\overline{a_{m-n}} \\
&= \sqrt{\frac{n+1}{m+1}}\overline{a_{m-n}}.
\end{aligned}
$$

情形(2): 如果 $m < n$, 那么

$$
\begin{aligned}
\langle CB_\phi Ce_{2n}, e_m\rangle &= \sqrt{m+1}\sqrt{n+1}\langle z^m, \sum_{j=0}^{\infty} b_j \overline{z}^j z^n\rangle \\
&= \sqrt{m+1}\sqrt{n+1}\sum_{j=0}^{\infty} \overline{b_j}\langle z^{m+j}, z^n\rangle \\
&= \sqrt{m+1}\sqrt{n+1}\frac{1}{n+1}\overline{b_{n-m}} \\
&= \sqrt{\frac{m+1}{n+1}}\overline{b_{n-m}}.
\end{aligned}
$$

$$
\begin{aligned}
\langle CB_\phi Ce_{2n+1}, e_m\rangle &= \langle Ce_m, B_\phi Ce_{2n+1}\rangle \\
&= \langle e_m, B_\phi \overline{e_{2n+1}}\rangle = \langle e_m, P_{L_a^2} M_\phi K(e_{2n+1})\rangle \\
&= \langle e_m, P_{L_a^2} M_\phi \overline{e_{n+1}}\rangle = \langle e_m, M_\phi \overline{e_{n+1}}\rangle = \langle e_m, \phi\overline{e_{n+1}}\rangle \\
&= \langle \sqrt{m+1}z^m, (\sum_{i=0}^{\infty} a_i z^i + \sum_{j=0}^{\infty} b_j \overline{z}^j)\sqrt{n+2}\overline{z}^{n+1}\rangle \\
&= \sqrt{m+1}\sqrt{n+1}\langle z^m, \sum_{i=0}^{\infty} a_i z^i \overline{z}^{n+1}\rangle + \langle z^m, \sum_{j=0}^{\infty} b_j \overline{z}^j \overline{z}^{n+1}\rangle \\
&= \sqrt{m+1}\sqrt{n+1}\sum_{i=0}^{\infty} \overline{a_i}\langle z^m, z^i \overline{z}^{n+1}\rangle \\
&= \frac{\sqrt{m+1}\sqrt{n+2}}{n+m+2}\overline{a_{m+n+1}}.
\end{aligned}
$$

其中 m 和 n 是非负整数. 因此, 可得到 $CB_\phi C$ 的矩阵形式如下:

$$
CB_\phi C = \begin{bmatrix}
\overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{2}}\overline{b_1} & \frac{1}{\sqrt{3}}\overline{a_2} & \frac{1}{\sqrt{3}}\overline{b_2} & \frac{1}{2}\overline{a_3} & \frac{1}{2}\overline{b_3}\cdots \\
\frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \overline{a_0} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{1}{\sqrt{2}}\overline{b_1} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \frac{1}{\sqrt{2}}\overline{b_2}\cdots \\
\frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \sqrt{\frac{2}{3}}\overline{a_1} & \frac{3}{5}\overline{a_4} & \overline{a_0} & \frac{1}{\sqrt{3}}\overline{a_5} & \frac{\sqrt{3}}{2}\overline{b_1}\cdots \\
\frac{1}{2}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \frac{1}{\sqrt{2}}\overline{a_2} & \frac{1}{\sqrt{3}}\overline{a_5} & \frac{\sqrt{3}}{2}\overline{a_1} & \frac{4}{7}\overline{a_6} & \overline{a_0}\cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.
$$

$$
B_\phi^* = \begin{bmatrix}
\overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{3}}\overline{a_2} & \frac{1}{2}\overline{a_3} & \cdots \\
\frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \cdots \\
\frac{1}{\sqrt{2}}\overline{b_1} & \overline{a_0} & \frac{\sqrt{3}}{2}\overline{a_1} & \frac{1}{\sqrt{2}}\overline{a_2} & \cdots \\
\frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{3}{5}\overline{a_0} & \frac{\sqrt{3}}{2}\overline{a_1} & \cdots \\
\frac{1}{2}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \frac{4}{7}\overline{a_6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.
$$

所以, 如果 $CB_\phi C = B_\phi^*$, 那么

$$
\begin{bmatrix}
\overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{2}}\overline{b_1} & \frac{1}{\sqrt{3}}\overline{a_2}\cdots \\
\frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \overline{a_0} & \frac{\sqrt{6}}{4}\overline{a_3}\cdots \\
\frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \sqrt{\frac{2}{3}}\overline{a_1} & \frac{3}{5}\overline{a_4}\cdots \\
\frac{1}{2}\overline{a}_3 & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \frac{1}{\sqrt{2}}\overline{a_2} & \frac{1}{\sqrt{3}}\overline{a_5}\cdots \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix}
=
\begin{bmatrix}
\overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{3}}\overline{a_2} & \frac{1}{2}\overline{a_3} & \cdots \\
\frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \cdots \\
\frac{1}{\sqrt{2}}\overline{b_1} & \overline{a_0} & \frac{\sqrt{3}}{2}\overline{a_1} & \frac{1}{\sqrt{2}}\overline{a_2} & \cdots \\
\frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{3}{5}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{bmatrix}.
$$

当 $n > m$ 时,

$$\sqrt{\frac{m+1}{n+1}}\overline{a_{n-m}} = \sqrt{\frac{m+1}{n+1}}\overline{b_{n-m}};$$

$$\sqrt{\frac{m+1}{n+1}}\overline{a_{n-m}} = \frac{\sqrt{m+1}\sqrt{m+2}}{m+n+2}\overline{a_{m+n+1}};$$

$$\sqrt{\frac{m+1}{n+1}}\overline{b_{n-m}} = \frac{\sqrt{m+1}\sqrt{n+2}}{m+n+2}\overline{a_{m+n+1}}.$$

当 $n \leq m$ 时,

$$\sqrt{\frac{n+1}{m+1}}\overline{a_{m-n}} = \sqrt{\frac{n+1}{m+1}}\overline{b_{m-n}};$$

$$\sqrt{\frac{n+1}{m+1}}\overline{a_{m-n}} = \frac{\sqrt{m+1}\sqrt{n+2}}{n+m+2}\overline{a_{m+n+1}};$$

$$\sqrt{\frac{n+1}{m+1}}\overline{b_{m-n}} = \frac{\sqrt{m+1}\sqrt{n+2}}{n+m+2}\overline{a_{m+n+1}}.$$

故对所有非负整数 m 和 n, 满足 $a_1 = a_2 = a_3 = \cdots = a_{n-m} = a_{m+n+1} \to \infty$. $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = \cdots = a_{n-m} = b_{n-m} = a_{m+n+1} \to \infty$, 当 $m, n \to \infty$. 矩阵中的每个元素都包含, 又因为在 $L^2(D)$ 中矩阵满足 $sup \sum_{i=0}^{\infty} \mid a_i \mid^2 < \infty$, $sup \sum_{j=0}^{\infty} \mid b_j \mid^2 < \infty$, 所以当 $m, n \to \infty$, $\lim\limits_{i\to\infty} \sum_{i=0}^{\infty} \mid a_i \mid^2 = 0$, $\lim\limits_{j\to\infty} \sum_{j=0}^{\infty} \mid b_j \mid^2 = 0$, 这意味着对于所有 i 和 j , $a_i = 0$ 且 $b_j = 0$. 因此我们得到了期望的结果 $\phi = 0$.

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