Advances in Applied Mathematics 应用数学进展, 2018, 7(4), 418-422
Published Online April 2018 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2018.74052

Hans汉斯

The Upper Bound of the Interval Edge-Coloring of the Generalized θ-Chain

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Received: Apr. 11th, 2018; accepted: Apr. 21st, 2018; published: Apr. 28th, 2018

Abstract

An edge-coloring of a graph G with colors $1,2,\cdots,t$ is an interval t-coloring if all colors are used, and the colors of edges incident to each vertex of G are distinct and form an interval of integer. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\aleph$. For any $G\in\aleph$, G has a interval t-coloring. $w(G)$ and $W(G)$ are the minimum and maximum numbers of t. A generalized θ-chain, denoted by $\theta_{m_1,m_2,\cdots,m_k}$, is a simple graph obtained by substituting $m_i\ge 2$ pairwise internally disjoint (v_{i-1},v_i)-paths for each edge $v_{i-1}v_i$ of the path $P=[v_0v_1\cdots v_k](k\ge 1)$, where $i=1,2,\cdots,k$. In this paper, we give a tight upper bound on $W(\theta_{m_1,m_2,\cdots,m_k})$.

Keywords

Interval Edge-Coloring, Upper Bound, Generalized θ-Graph, Generalized θ-Chain

广义θ-链的区间边着色的上界

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收稿日期：2018年4月11日；录用日期：2018年4月21日；发布日期：2018年4月28日

摘　要

图G的一个用了颜色$1,2,\cdots,t$的边着色称为区间t-着色，如果所有t种颜色都被用到，并且关联于G的同一

文章引用: 陈勋. 广义 θ-链的区间边着色的上界[J]. 应用数学进展, 2018, 7(4): 418-422.
DOI: 10.12677/aam.2018.74052

个顶点的边上颜色各不相同且这些颜色构成了一个连续的整数区间。**G称作是可区间着色的，如果对某个正整数t，G有一个区间t-着色。所有可区间着色的图构成的集合记作$\aleph$。对图$G\in\aleph$，使得G有一个区间t-着色的t的最小值和最大值分别记作$w(G)$和$W(G)$。广义θ-链，记作$\theta_{m_1,m_2,\cdots,m_k}$，是把路$P=[v_0v_1\cdots v_k](k\geq 1)$的每一条边$v_{i-1}v_i$用$m_i\geq 2$条两两内部不交的$(v_{i-1},v_i)$-路替换掉而得到的简单图，这里$i=1,2,\cdots,k$。在本文中，我们给出了$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)$的一个紧的上界。**

关键词

区间边着色，上界，广义θ-图，广义θ-链

1. 引言

$V(G)$和$E(G)$分别表示图G的顶点集和边集，$d_G(v)$表示顶点$v\in V(G)$在图G中的度，$\Delta(G)$表示图G的最大度，$\chi'(G)$表示图G的边色数，P_n表示n个顶点的路。

如果α是图G的一个正常边着色，那么与$v\in V(G)$关联的边上的颜色构成的集合，记作$S(v,\alpha)$。我们用$\underline{S}(v,\alpha)$和$\overline{S}(v,\alpha)$分别表示$S(v,\alpha)$的最小色值和最大色值。

图G的一个用了颜色$1,2,\cdots,t$的正常边着色α称为区间t-着色，如果所有t种颜色都被用到，并且对任意的$v\in V(G)$，$S(v,\alpha)$是连续的。图G称为是可区间着色的，如果对某个正整数t，G有一个区间t-着色。所有可区间着色的图构成的集合记作$\aleph$。对图$G\in\aleph$，使得G有一个区间t-着色的t的最小值和最大值分别记作$w(G)$和$W(G)$。

区间边着色的概念是由 Asratian 和 Kamalian 在[1] [2]中提出来的，并且证明了如果$G\in\aleph$，那么$\chi'(G)=\Delta(G)$。在[3]中，Kamalian 证明了如果$G\in\aleph$是一个连通图，那么$W(G)\leq 2|V(G)|-3$。这个上界被 Giaro 等在[4]中改进了，他们证明了如果$G\in\aleph$是一个至少有 3 个顶点的连通图，那么$W(G)\leq 2|V(G)|-4$。对某些图类来说，$W(G)$的上界还得到了进一步的改进，其中包括不含三角形的图[1] [2]和可平面图[5]。

广义θ-图是由$m\geq 2$条两两内部不交的(u,v)-路构成的简单图，记作$\theta_m=[u,\theta_m,v]$。广义θ-链是把路$P=[v_0v_1\cdots v_k](k\geq 1)$的每一条边$v_{i-1}v_i$（$i=1,2,\cdots,k$）用$m_i\geq 2$条两两内部不交的$(v_{i-1},v_i)$-路替换掉得到的简单图，记作$\theta_{m_1,m_2,\cdots,m_k}=\left[v_0,\theta_{m_1},v_1,\theta_{m_2},\cdots,\theta_{m_k},v_k\right]$。在本文中，我们给出了$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)$的一个紧的上界。

2. $W\left(\theta_{m_1,m_2,\cdots,m_k}\right)$的一个上界

令$\theta_{m_1,m_2,\cdots,m_k}=\left[v_0,\theta_{m_1},v_1,\theta_{m_2},\cdots,\theta_{m_k},v_k\right]$，$P_{n_{ij}}\left(i=1,2,\cdots,k;j=1,2,\cdots,m_i\right)$是$\theta_{m_i}$中两两内部不交的$(u_{i-1},v_i)$-路。对每一个$i\in\{1,2,\cdots,k\}$，令

$$L\left(\theta_{m_i}\right)=\max\left\{\left|E\left(P_{n_{ij}}\right)\right| \mid j=1,2,\cdots,m_i\right\},\quad l\left(\theta_{m_i}\right)=\min\left\{\left|E\left(P_{n_{ij}}\right)\right| \mid j=1,2,\cdots,m_i\right\}$$

为了方便，我们记 $L_i\left(\theta_{m_1,m_2,\cdots,m_k}\right)$ 为 $\left\{L\left(\theta_{m_i}\right)|i=1,2,\cdots,k\right\}$ 中第 i 大的数，$l_i\left(\theta_{m_1,m_2,\cdots,m_k}\right)$ 为 $\left\{l\left(\theta_{m_i}\right)|i=1,2,\cdots,k\right\}$ 中第 i 大的数。

引理 2.1：设 α 是可区间着色图 G 的一个区间 t-着色，$P_s=\left(v_1,v_2,\cdots,v_s\right)$ 是 G 中的一条路，其中 $1\in S\left(v_1,\alpha\right)$ 且 $t\in S\left(v_s,\alpha\right)$。那么 $t\le 1+\sum_{k=1}^{s}\left(d_G\left(v_k\right)-1\right)$，等号成立当且仅当

$$\alpha\left(v_iv_{i+1}\right)=\overline{S}\left(v_i,\alpha\right)=\underline{S}\left(v_{i+1},\alpha\right)=1+\sum_{k=1}^{i}\left(d_G\left(v_k\right)-1\right),\quad i=1,2,\cdots,s-1。$$

证明：因为 $\underline{S}\left(v_1,\alpha\right)=1$ 且 $\alpha\left(v_1v_2\right)\le\overline{S}\left(v_1,\alpha\right)$，我们有

$$\alpha\left(v_1v_2\right)-1\le\overline{S}\left(v_1,\alpha\right)-\underline{S}\left(v_1,\alpha\right)=d_G\left(v_1\right)-1,\tag{1}$$

等号成立当且仅当 $\alpha\left(v_1v_2\right)=\overline{S}\left(v_1,\alpha\right)=d_G\left(v_1\right)$。因为 $\overline{S}\left(v_s,\alpha\right)=t$ 且 $\underline{S}\left(v_s,\alpha\right)\le\alpha\left(v_{s-1}v_s\right)$，我们有

$$t-\alpha\left(v_{s-1}v_s\right)\le\overline{S}\left(v_s,\alpha\right)-\underline{S}\left(v_s,\alpha\right)=d_G\left(v_s\right)-1,\tag{2}$$

等号成立当且仅当 $\alpha\left(v_{s-1}v_s\right)=\underline{S}\left(v_s,\alpha\right)$。因为 $\underline{S}\left(v_{i+1},\alpha\right)\le\alpha\left(v_iv_{i+1}\right)\le\overline{S}\left(v_i,\alpha\right)$，$i=1,2,\cdots,s-1$，我们有

$$\alpha\left(v_iv_{i+1}\right)-\alpha\left(v_{i-1}v_i\right)\le\overline{S}\left(v_i,\alpha\right)-\underline{S}\left(v_i,\alpha\right)=d_G\left(v_i\right)-1,\quad i=1,2,\cdots,s-1\tag{3}$$

等号成立当且仅当 $\alpha\left(v_iv_{i+1}\right)=\overline{S}\left(v_i,\alpha\right)=\alpha\left(v_{i-1}v_i\right)+d_G\left(v_i\right)-1$ 且 $\alpha\left(v_{i-1}v_i\right)=\underline{S}\left(v_i,\alpha\right)$。

由(1)-(3)，我们可以得到

$$t-1=\left(\alpha\left(v_1v_2\right)-1\right)+\sum_{k=2}^{s-1}\left(\alpha\left(v_iv_{i+1}\right)-\alpha\left(v_{i-1}v_i\right)\right)+\left(t-\alpha\left(v_{s-1}v_s\right)\right)\le\sum_{k=1}^{s}\left(d_G\left(v_k\right)-1\right),$$

等号成立当且仅当 $\alpha\left(v_iv_{i+1}\right)=\overline{S}\left(v_i,\alpha\right)=\underline{S}\left(v_{i+1},\alpha\right)=1+\sum_{k=1}^{i}\left(d_G\left(v_k\right)-1\right)$，$i=1,2,\cdots,s-1$。

证毕。

定理 2.3：若广义 θ-链 $\theta_{m_1,m_2,\cdots,m_k}=\left[v_0,\theta_{m_1},v_1,\theta_{m_2},\cdots,\theta_{m_k},v_k\right]$ 可区间边着色的，那么

$$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\le\sum_{i=1}^{2}L_i\left(\theta_{m_1,m_2,\cdots,m_k}\right)+\sum_{i=1}^{k-2}l_i\left(\theta_{m_1,m_2,\cdots,m_k}\right)+\sum_{i=1}^{k}\left(m_i-1\right)+\sum_{i=2}^{k-1}\left(m_i-1\right).$$

证明：设 α 是 $\theta_{m_1,m_2,\cdots,m_k}$ 的一个区间 $W\left(\theta_{m_1,m_2,\cdots,m_k}\right)$-着色。那么存在两条边 $e',e''\in E\left(\theta_{m_1,m_2,\cdots,m_k}\right)$ 使得 $\alpha\left(e'\right)=1$ 且 $\alpha\left(e''\right)=W\left(\theta_{m_1,m_2,\cdots,m_k}\right)$。我们分以下两种情况讨论。

Case 1. $e',e''\in E\left(\theta_{m_r}\right)$，$r\in\left\{1,2,\cdots,k\right\}$。

首先，我们假设 $e',e''\in E\left(P_{n_{rj}}\right)$，$j\in\left\{1,2,\cdots,m_r\right\}$。令 $P_{n_{rj}}=\left(u_1,u_2,\cdots,u_{n_{rj}}\right)$，$e'=u_{s_1}u_{s_1+1}$，$e''=u_{s_2}u_{s_2+1}$，这里 $1\le s_1<s_2\le n_{rj}-1$ 且 $\left\{u_1,u_{n_{rj}}\right\}=\left\{v_{r-1},v_r\right\}$。那么 $1\in S\left(u_{s_1+1},\alpha\right)$，$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\in S\left(u_{s_2},\alpha\right)$。由引理 2.1，我们有

$$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\le 1+\sum_{i=s_1+1}^{s_2}\left(d\left(u_i\right)-1\right)\le 1+\sum_{i=2}^{n_{rj}-1}d\left(u_i\right)-1。\tag{4}$$

因为 $d\left(u_2\right)=\cdots=d\left(u_{n_{rj}-1}\right)=2$ 且 $\left|E\left(P_{n_{rj}}\right)\right|=n_{rj}-1$，由(4)我们有

$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\le n_{rj}-1\le L\left(\theta_{m_r}\right)\le L_1\left(\theta_{m_1,m_2,\cdots,m_k}\right)$，因此结论成立。

其次，我们假设 $e'\in E\left(P_{n_{rp}}\right)$ 且 $e''\in E\left(P_{n_{rq}}\right)$，这里 $q\ne p\in\left\{1,2,\cdots,m_r\right\}$。设 $P_{n_{rp}}=\left(x_1,x_2,\cdots,x_{n_{rp}}\right)$，$P_{n_{rq}}=\left(y_1,y_2,\cdots,y_{n_{rq}}\right)$，$e'=x_tx_{t+1}$，$e''=y_dy_{d+1}$，这里 $t\in\left\{1,2,\cdots,n_{rp}-1\right\}$，$d\in\left\{1,2,\cdots,n_{rq}-1\right\}$，$x_1=y_1=v_{r-1}$，

$x_{n_{rp}}=y_{n_{rq}}=v_r$。我们考虑以下两条路:

$$P^1=\left(x_t,x_{t-1},\cdots,x_1,y_2,y_3,\cdots,y_d\right),\quad P^2=\left(y_{d+1},y_{d+2},\cdots,y_{n_{rq}},x_{n_{rp}-1},x_{n_{rp}-2},\cdots,x_{t+1}\right).$$

显然，$\left|E\left(P^1\right)\right|+\left|E\left(P^2\right)\right|+2=\left|E\left(P_{n_{rp}}\right)\right|+\left|E\left(P_{n_{rq}}\right)\right|\le 2L\left(\theta_{m_r}\right)$，因此$\left|E\left(P^1\right)\right|+\left|E\left(P^2\right)\right|\le 2\left(L\left(\theta_{m_r}\right)-1\right)$。我们能断言$\left|E\left(P^1\right)\right|\le L\left(\theta_{m_r}\right)-1$或者$\left|E\left(P^2\right)\right|\le L\left(\theta_{m_r}\right)-1$，否则$\left|E\left(P^1\right)\right|+\left|E\left(P^2\right)\right|\ge 2L\left(\theta_{m_r}\right)$，矛盾。不失一般性，设$\left|E\left(P^1\right)\right|\le L\left(\theta_{m_r}\right)-1$，那么$\left|V\left(P^1\right)\right|\le L\left(\theta_{m_r}\right)$。注意到$1\in S\left(x_t,\alpha\right)$且$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\in S\left(y_d,\alpha\right)$。由引理 2.1，我们有

$$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\le 1+\sum_{u\in V\left(P^1\right)}\left(d\left(u\right)-1\right). \tag{5}$$

因为$d\left(x_1\right)=d\left(v_{r-1}\right)=m_{r-1}+m_r$，且对任意$u\in V\left(P^1\right)\setminus\left\{x_1\right\}$有$d\left(u\right)=2$，由(5)，我们有$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\le m_{r-1}+m_r+\left|V\left(P^1\right)\right|-1\le L\left(\theta_{m_r}\right)+m_{r-1}+m_r-1\le L_1\left(\theta_{m_1,m_2,\cdots,m_k}\right)+m_{r-1}+m_r-1$，因此结论成立。

Case 2. $e'\in E\left(\theta_{m_r}\right)$，$e''\in E\left(\theta_{m_s}\right)$，$s\ne r\in\{1,2,\cdots,k\}$.

不失一般性，设$1\le r<s\le k$，$\left|E\left(P_{n_{i1}}\right)\right|=L\left(\theta_{m_i}\right)$且$\left|E\left(P_{n_{im_i}}\right)\right|=l\left(\theta_{m_i}\right)(i=1,2,\cdots,k)$，$e'\in E\left(P_{n_{rp}}\right)$且$e''\in E\left(P_{n_{sq}}\right)$，这里$p\in\{1,2,\cdots,m_r\}$且$q\in\{1,2,\cdots,m_s\}$。设$P_{n_{rp}}=\left(x_1,x_2,\cdots,x_{n_{rp}}\right)$，$P_{n_{sq}}=\left(y_1,y_2,\cdots,y_{n_{sq}}\right)$，$e'=x_tx_{t+1}$，$e''=y_dy_{d+1}$，这里$t\in\left\{1,2,\cdots,n_{rp}-1\right\}$，$d\in\left\{1,2,\cdots,n_{sq}-1\right\}$，$x_1=v_{r-1}$，$x_{n_{rp}}=v_r$，$y_1=v_{s-1}$，$y_{n_{sq}}=v_s$。那么

$$P^1=\left(x_{t+1},x_{t+2},\cdots,x_{n_{rp}}\right)\cup P_{n_{(r+1)m_{r+1}}}\cup P_{n_{(r+2)m_{r+2}}}\cup\cdots\cup P_{n_{(s-1)m_{s-1}}}\cup\left(y_1,y_2,\cdots,y_d\right)$$

是$\theta_{m_1,m_2,\cdots,m_k}$中的一条路满足$1\in S\left(x_{t+1},\alpha\right)$且$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\in S\left(y_d,\alpha\right)$。由引理 2.1，可得

$$W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\le 1+\sum_{u\in V\left(P^1\right)}\left(d\left(u\right)-1\right). \tag{6}$$

令

$$P^2=\begin{cases}\left(P_{n_{1m_1}}\setminus\{v_0\}\right)\cup\bigcup\limits_{i=2}^{r-1}P_{n_{im_i}}\cup P_{n_{r1}}\cup\bigcup\limits_{i=r+1}^{s-1}P_{n_{im_i}}\cup P_{n_{s1}}\cup\bigcup\limits_{i=s+1}^{k-1}P_{n_{im_i}}\cup\left(P_{n_{km_k}}\setminus\{v_k\}\right), & 1<r<s<k,\\ \left(P_{n_{11}}\setminus\{v_0\}\right)\cup\bigcup\limits_{i=r+1}^{s-1}P_{n_{im_i}}\cup P_{n_{s1}}\cup\bigcup\limits_{i=s+1}^{k-1}P_{n_{im_i}}\cup\left(P_{n_{km_k}}\setminus\{v_k\}\right), & 1=r<s<k,\\ \left(P_{n_{1m_1}}\setminus\{v_0\}\right)\cup\bigcup\limits_{i=2}^{r-1}P_{n_{im_i}}\cup P_{n_{r1}}\cup\bigcup\limits_{i=r+1}^{s-1}P_{n_{im_i}}\cup\left(P_{n_{k1}}\setminus\{v_k\}\right), & 1<r<s=k.\end{cases}$$

显然，$V\left(P^1\right)\subseteq V\left(P^2\right)$，表明$\sum_{u\in V\left(P^1\right)}\left(d\left(u\right)-1\right)\le\sum_{u\in V\left(P^2\right)}\left(d\left(u\right)-1\right)$。因为$d\left(v_i\right)=m_i+m_{i+1}\quad(i=1,2,\cdots,k-1)$，且对任意$u\in V\left(P^2\right)\setminus\left\{v_1,v_2,\cdots,v_{k-1}\right\}$有$d\left(u\right)=2$，$L\left(\theta_{m_i}\right)=\left|V\left(P_{n_{i1}}\right)\right|-1\quad(i=1,2,\cdots,k)$，由(6)，我们有

$$\begin{aligned}&W\left(\theta_{m_1,m_2,\cdots,m_k}\right)\le 1+\sum_{u\in V\left(P^1\right)}\left(d\left(u\right)-1\right)\le 1+\sum_{u\in V\left(P^2\right)}\left(d\left(u\right)-1\right)\\&=1+\sum_{1\le i\le k-1}\left(d\left(v_i\right)-1\right)+\sum_{\substack{u\in V\left(P_{n_{im_i}}\right)\setminus\{v_{i-1},v_i\}\\ i\in\{1,2,\cdots,k\}\setminus\{r,s\}}}\left(d\left(u\right)-1\right)+\sum_{\substack{u\in V\left(P_{n_{i1}}\right)\setminus\{v_{i-1},v_i\}\\ i\in\{r,s\}}}\left(d\left(u\right)-1\right)\\&=1+\sum_{1\le i\le k-1}\left(m_i+m_{i+1}-1\right)+\sum_{i\in\{1,2,\cdots,k\}\setminus\{r,s\}}\left(L\left(\theta_{m_i}\right)-1\right)+\sum_{i\in\{r,s\}}\left(L\left(\theta_{m_i}\right)-1\right)\end{aligned}$$

$$= \sum_{1\le i\le k-1}(m_i-1)+\sum_{1\le i\le k-1}(m_{i+1}-1)+\sum_{i\in\{1,2,\cdots,k\}\setminus\{r,s\}} l(\theta_{m_i})+\sum_{i\in\{r,s\}} L(\theta_{m_i})$$

$$\le \sum_{1\le i\le 2} L_i(\theta_{m_1,m_2,\cdots,m_k})+\sum_{1\le i\le k-2} l_i(\theta_{m_1,m_2,\cdots,m_k})+\sum_{1\le i\le k-1}(m_i-1)+\sum_{2\le i\le k}(m_i-1)$$

$$= \sum_{1\le i\le 2} L_i(\theta_{m_1,m_2,\cdots,m_k})+\sum_{1\le i\le k-2} l_i(\theta_{m_1,m_2,\cdots,m_k})+\sum_{1\le i\le k}(m_i-1)+\sum_{2\le i\le k-1}(m_i-1)$$

因此结论成立。

证毕。

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