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General Expression Analysis of the Hamiltonian Operator and Its Formula

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Abstract

The Hamiltonian operator ∇ and the common expressions such as the Laplacian operator, gradient, divergence, and curl generated by it are not the same in different curve coordinate systems. This paper defines a three-dimensional orthogonal curve coordinate system (u_1, u_2, u_3), introducing coordinate factor h_1, h_2, h_3, deriving the general form of ∇, $\nabla\phi$, $\nabla\cdot A$, $\nabla\times A$, ∇^2, and the general expression of Poisson equation as well as Laplace equation.

Keywords

Hamiltonian Operators, Laplacian Operator, Gradient, Divergence, Curl

哈密顿算符及其一般运算表达式的分析

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摘 要

哈密顿算符∇及其产生的拉普拉斯算符、梯度、散度和旋度常见运算式在不同曲线坐标系中具体表达式不相同。本文通过定义一个三维正交曲线坐标系(u_1, u_2, u_3)，引入坐标因子h_1、h_2、h_3，推导得到了关于∇、$\nabla\phi$、$\nabla\cdot A$、$\nabla\times A$、∇^2的一般形式及Poisson方程和Laplace方程的一般表达式。

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关键词

哈密顿算符，拉普拉斯算符，梯度，散度，旋度

1. 引言

哈密顿算符 ∇ 是关于空间的一阶偏微分算子，在数学和物理学中发挥着重要作用。由 ∇ 可以产生一系列的偏微分方程，如泊松方程、拉普拉斯方程等，这些方程在电磁学、热传导等方面经常用到[1] [2] [3]。∇ 及其产生的梯度、散度和旋度等常见运算式在笛卡尔坐标系、柱坐标系、球坐标系、和极坐标系下，其具体表达形式是不一样的，能不能给各种运算式找到一个通用的一般表达式来统一呢？本文针对这个问题做了一些分析，在三维正交曲线坐标系 (u_1,u_2,u_3) 下，得到了几个包含哈密顿算符运算式的一般表达式。

2. 正交曲线坐标系

考虑一个由三维正交曲线坐标系 (u_1,u_2,u_3) [4]组成的空间，其中一个无限小的体积元可由 u_1，$u_1+\mathrm{d}u_1$，u_2，$u_2+\mathrm{d}u_2$，u_3，$u_3+\mathrm{d}u_3$ 相围得到。一般通过 u_1、u_2、u_3 乘以坐标因子 h_1，h_2，h_3 来表示三个方向的距离，h_1，h_2，h_3 是 (u_1,u_2,u_3) 的函数。

如图 1 所示，定义三个线元：

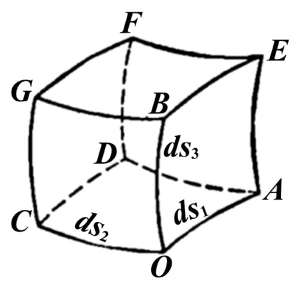

Figure 1. The map of generalized coordinates volume element
图 1. 广义坐标系体积元图

$$\mathrm{d}s_1=h_1\mathrm{d}u_1,\quad \mathrm{d}s_2=h_2\mathrm{d}u_2,\quad \mathrm{d}s_3=h_3\mathrm{d}u_3 \tag{1.1}$$

体积元 $\mathrm{d}V=\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3=h_1h_2h_3\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3$，则哈密顿算符可以表示为：

$$\nabla=\frac{\partial}{\partial s_1}\boldsymbol{e}_1+\frac{\partial}{\partial s_2}\boldsymbol{e}_2+\frac{\partial}{\partial s_3}\boldsymbol{e}_3=\frac{\partial}{h_1\partial u_1}\boldsymbol{e}_1+\frac{\partial}{h_2\partial u_2}\boldsymbol{e}_2+\frac{\partial}{h_3\partial u_3}\boldsymbol{e}_3 \tag{1.2}$$

$\boldsymbol{e}_1$、$\boldsymbol{e}_2$、$\boldsymbol{e}_3$ 分别是空间变量 (u_1,u_2,u_3) 的单位矢量。

1) 一标量 ϕ 在正交曲线坐标系 (u_1,u_2,u_3) 连续，一阶微分存在，则根据梯度的定义式，ϕ 的梯度分量：

$$\frac{\partial\phi}{\partial s_1}=\frac{\partial\phi}{h_1\partial u_1},\quad \frac{\partial\phi}{\partial s_2}=\frac{\partial\phi}{h_2\partial u_2},\quad \frac{\partial\phi}{\partial s_3}=\frac{\partial\phi}{h_3\partial u_3} \tag{1.3}$$

$$\nabla\phi=\left(\frac{\partial\phi}{h_1\partial u_1}\right)\boldsymbol{e}_1+\left(\frac{\partial\phi}{h_2\partial u_2}\right)\boldsymbol{e}_2+\left(\frac{\partial\phi}{h_3\partial u_3}\right)\boldsymbol{e}_3 \tag{1.4}$$

2) 一矢量 $\boldsymbol{A}$ 在正交曲线坐标系 (u_1,u_2,u_3) 连续，一阶微分存在，那么矢量 $\boldsymbol{A}$ 穿过面 OCGB 和 ADFE 的向外通量为：

$$\frac{\partial}{\partial s_1}\left(A_1\mathrm{d}s_2\mathrm{d}s_3\right)\mathrm{d}s_1=\frac{\partial}{\partial u_1}\left(h_2h_3A_1\right)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3=\frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}\left(h_2h_3A_1\right)\mathrm{d}V \tag{1.5}$$

用同样的方法可以得到穿过其他 4 个平面的通量，则根据散度的定义，$\boldsymbol{A}$ 的散度：

$$\mathrm{div}\boldsymbol{A}=\nabla\cdot\boldsymbol{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(h_2h_3A_1\right)+\frac{\partial}{\partial u_2}\left(h_3h_1A_2\right)+\frac{\partial}{\partial u_3}\left(h_1h_2A_3\right)\right] \tag{1.6}$$

3) 矢量 $\boldsymbol{A}$ 是 (u_1,u_2,u_3) 的函数，在正交曲线坐标下连续可微，A_1、A_2、A_3 是沿着 u_1、u_2、u_3 三个变量方向上的分量，由 Stokes 定理，在面 OADC 上 A_1 沿着边 OA 和 DC 的线积分等于：

$$\left[h_1\left(u_1,u_2\right)A_1\left(u_1,u_2\right)-h_1\left(u_1,u_2+\mathrm{d}u_2\right)A_1\left(u_1,u_2+\mathrm{d}u_2\right)\right]\mathrm{d}u_1=-\frac{\partial\left(h_1A_1\right)}{\partial u_2}\mathrm{d}u_1\mathrm{d}u_2 \tag{1.7}$$

沿着边 AD 和 CO 的线积分等于

$$\left[h_2\left(u_1+\mathrm{d}u_1,u_2\right)A_2\left(u_1+\mathrm{d}u_1,u_2\right)-h_2\left(u_1,u_2\right)A_2\left(u_1,u_2\right)\right]\mathrm{d}u_2=\frac{\partial\left(h_2A_2\right)}{\partial u_1}\mathrm{d}u_1\mathrm{d}u_2 \tag{1.8}$$

根据 Stokes 定理，$\boldsymbol{A}$ 沿着面 OADC 边界上的环路积分等于 $\nabla\times\boldsymbol{A}$ 穿过平面 OADC 的通量，即：

$$\left(\nabla\times\boldsymbol{A}\right)_3=\frac{1}{h_1h_2}\left[\frac{\partial\left(h_2A_2\right)}{\partial u_1}-\frac{\partial\left(h_1A_1\right)}{\partial u_2}\right] \tag{1.9}$$

对于另外两个面 OBGC 和 OAEB 也可以得到

$$\left(\nabla\times\boldsymbol{A}\right)_1=\frac{1}{h_2h_3}\left[\frac{\partial\left(h_3A_3\right)}{\partial u_2}-\frac{\partial\left(h_2A_2\right)}{\partial u_3}\right] \tag{1.10}$$

$$\left(\nabla\times\boldsymbol{A}\right)_2=\frac{1}{h_3h_1}\left[\frac{\partial\left(h_1A_1\right)}{\partial u_3}-\frac{\partial\left(h_3A_3\right)}{\partial u_1}\right] \tag{1.11}$$

所以，矢量 $\boldsymbol{A}$ 的旋度为

$$\begin{aligned}\mathrm{curl}\boldsymbol{A}&=\nabla\times\boldsymbol{A}\\&=\frac{1}{h_2h_3}\left[\frac{\partial\left(h_3A_3\right)}{\partial u_2}-\frac{\partial\left(h_2A_2\right)}{\partial u_3}\right]\boldsymbol{e}_1+\frac{1}{h_3h_1}\left[\frac{\partial\left(h_1A_1\right)}{\partial u_3}-\frac{\partial\left(h_3A_3\right)}{\partial u_1}\right]\boldsymbol{e}_2\\&\quad+\frac{1}{h_1h_2}\left[\frac{\partial\left(h_2A_2\right)}{\partial u_1}-\frac{\partial\left(h_1A_1\right)}{\partial u_2}\right]\boldsymbol{e}_3\end{aligned} \tag{1.12}$$

3. 几例含哈密顿算符的运算式

3.1. Poisson 方程和 Laplace 方程

标量 ϕ 在正交曲线坐标系 (u_1,u_2,u_3) 连续可微，存在二阶偏导数，且矢量 $\boldsymbol{A}=\varepsilon\nabla\phi$（如电势和电场强度），那么 $\nabla\cdot\boldsymbol{A}$ 等于什么呢？将(1.4)式中 3 个分量乘以 ε 代替(1.6)式中的 A_1、A_2、A_3 有：

$$\nabla\cdot\boldsymbol{A}=\nabla\cdot\varepsilon\nabla\phi=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\varepsilon\frac{\partial\phi_1}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\varepsilon\frac{\partial\phi_2}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\varepsilon\frac{\partial\phi_3}{\partial u_3}\right)\right] \tag{2.1}$$

若 $\nabla\cdot\boldsymbol{A}=f$ ，且 $f\neq 0$ 则(2.1)式称 Poisson 方程；若 $f\neq 0$ 则称为 Laplace 方程，令 $\varepsilon=1$ ，可以得到拉普拉斯算符：

$$\nabla^2=\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right] \tag{2.2}$$

3.2. $\nabla\cdot(\mu\boldsymbol{A})=(\nabla\mu)\cdot\boldsymbol{A}+\mu(\nabla\cdot\boldsymbol{A})$

$\mu(u_1,u_2,u_3)$ 连续可微，由(1.4)和(1.6)式可得到：

$$\begin{aligned}\nabla\cdot(\mu\boldsymbol{A})&=(\nabla\mu)\cdot\boldsymbol{A}+\mu(\nabla\cdot\boldsymbol{A})\\&=\left[\left(\frac{\partial\mu}{h_1\partial u_1}\right)A_1+\left(\frac{\partial\mu}{h_2\partial u_2}\right)A_2+\left(\frac{\partial\mu}{h_3\partial u_3}\right)A_3\right]\\&\quad+\frac{\mu}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1)+\frac{\partial}{\partial u_2}(h_3h_1A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]\end{aligned} \tag{2.3}$$

3.3. $\nabla^2\boldsymbol{A}=(\nabla^2A_1)\boldsymbol{e}_1+(\nabla^2A_2)\boldsymbol{e}_2+(\nabla^2A_3)\boldsymbol{e}_3$

由(2.2)式可得：

$$\begin{aligned}\nabla^2\boldsymbol{A}&=(\nabla^2A_1)\boldsymbol{e}_1+(\nabla^2A_2)\boldsymbol{e}_2+(\nabla^2A_3)\boldsymbol{e}_3\\&=\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial A_1}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial A_1}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial A_1}{\partial u_3}\right)\right]\boldsymbol{e}_1\\&\quad+\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial A_2}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial A_2}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial A_2}{\partial u_3}\right)\right]\boldsymbol{e}_2\\&\quad+\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial A_3}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial A_3}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial A_3}{\partial u_3}\right)\right]\boldsymbol{e}_3\end{aligned} \tag{2.4}$$

3.4. $\nabla\times(\nabla\times\boldsymbol{A})=\nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}$

由(1.4)，(1.6)，(2.4)三式可以得到

$$\begin{aligned}&\nabla\times(\nabla\times\boldsymbol{A})\\&=\nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}\\&=\frac{\partial}{h_1\partial u_1}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_1h_2A_1)+\frac{\partial}{\partial u_2}(h_3h_1A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]\right\}\boldsymbol{e}_1\\&\quad+\frac{\partial}{h_2\partial u_2}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_1h_2A_1)+\frac{\partial}{\partial u_2}(h_3h_1A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]\right\}\boldsymbol{e}_2\\&\quad+\frac{\partial}{h_3\partial u_3}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_1h_2A_1)+\frac{\partial}{\partial u_2}(h_3h_1A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]\right\}\boldsymbol{e}_3\end{aligned}$$

$$-\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial A_1}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial A_1}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial A_1}{\partial u_3}\right)\right]\boldsymbol{e}_1$$
$$-\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial A_2}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial A_2}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial A_2}{\partial u_3}\right)\right]\boldsymbol{e}_2 \tag{2.5}$$
$$-\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial A_3}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial A_3}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial A_3}{\partial u_3}\right)\right]\boldsymbol{e}_3$$

3.5. 动量算符 $-i\hbar\nabla$ 与 Schrödinger 方程

由(1.2)式可知动量算符 $-i\hbar\nabla=-i\hbar\left(\boldsymbol{e}_1\frac{\partial}{h_1\partial u_1}+\boldsymbol{e}_2\frac{\partial}{h_2\partial u_2}+\boldsymbol{e}_3\frac{\partial}{h_3\partial u_3}\right)$，自由粒子波函数 ψ 的波动方程为

$$-i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]+V(u_1,u_2,u_3)\psi \tag{2.6}$$

4. 在笛卡尔坐标系、柱坐标系和球坐标系的具体形式

4.1. 笛卡尔坐标系

在笛卡尔坐标系中，坐标因子 $h_1=1$，$h_2=1$，$h_3=1$，$\mathrm{d}s_1=x$，$\mathrm{d}s_2=y$，$\mathrm{d}s_3=z$，则：

$$\nabla=\frac{\partial}{\partial s_1}\boldsymbol{e}_1+\frac{\partial}{\partial s_2}\boldsymbol{e}_2+\frac{\partial}{\partial s_3}\boldsymbol{e}_3=\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}$$

$$\nabla\phi=\left(\frac{\partial\phi}{\partial x}\right)\boldsymbol{i}+\left(\frac{\partial\phi}{\partial y}\right)\boldsymbol{j}+\left(\frac{\partial\phi}{\partial z}\right)\boldsymbol{k}$$

$$\mathrm{div}\boldsymbol{A}=\nabla\cdot\boldsymbol{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

$$\mathrm{curl}\boldsymbol{A}=\nabla\times\boldsymbol{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\boldsymbol{k}$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

4.2. 柱坐标系

在柱坐标系中，坐标因子 $h_1=1$，$h_2=\rho$，$h_3=1$，$\mathrm{d}s_1=\mathrm{d}\rho$，$\mathrm{d}s_2=\rho\mathrm{d}\varphi$，$\mathrm{d}s_3=\mathrm{d}z$ 则：

$$\nabla=\frac{\partial}{\partial s_1}\boldsymbol{e}_1+\frac{\partial}{\partial s_2}\boldsymbol{e}_2+\frac{\partial}{\partial s_3}\boldsymbol{e}_3=\frac{\partial}{\partial\rho}\boldsymbol{e}_\rho+\frac{1}{\rho}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi+\frac{\partial}{\partial z}\boldsymbol{e}_z$$

$$\nabla\phi=\left(\frac{\partial\phi}{\partial\rho}\right)\boldsymbol{e}_\rho+\frac{1}{\rho}\left(\frac{\partial\phi}{\partial\varphi}\right)\boldsymbol{e}_\varphi+\left(\frac{\partial\phi}{\partial z}\right)\boldsymbol{e}_z$$

$$\mathrm{div}\boldsymbol{A}=\nabla\cdot\boldsymbol{A}=\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$$

$$\mathrm{curl}\boldsymbol{A}=\nabla\times\boldsymbol{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial z}\right)\boldsymbol{e}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\boldsymbol{e}_\varphi+\frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial\rho}-\frac{\partial A_\rho}{\partial\varphi}\right)\boldsymbol{e}_z$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

4.3. 球坐标系

球坐标系中，坐标因子 $h_1 = 1$， $h_2 = r$， $h_3 = r\sin\theta$， $\mathrm{d}s_1 = \mathrm{d}r$， $\mathrm{d}s_2 = r\mathrm{d}\theta$， $\mathrm{d}s_3 = r\sin\theta\mathrm{d}\varphi$，则：

$$\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi$$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}\right)\boldsymbol{e}_r + \frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right)\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial\phi}{\partial\varphi}\right)\boldsymbol{e}_\varphi$$

$$\mathrm{div}\boldsymbol{A} = \nabla\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(A_\theta\sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\mathrm{curl}\boldsymbol{A} = \nabla\times\boldsymbol{A} = \frac{1}{r\sin\theta}\left(\frac{\partial\left(A_\varphi\sin\theta\right)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\boldsymbol{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial\left(rA_\varphi\right)}{\partial r}\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\boldsymbol{e}_\varphi$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

5. 结论

通过定义一个三维正交曲线坐标系 (u_1, u_2, u_3) 组成的空间，以及用 u_1、u_2、u_3 乘以坐标因子 h_1、h_2、h_3 来表示三个方向的线长，h_1、h_2、h_3 是 (u_1, u_2, u_3) 的函数。定义三个线元 $\mathrm{d}s_1 = h_1\mathrm{d}u_1$，$\mathrm{d}s_2 = h_2\mathrm{d}u_2$，$\mathrm{d}s_3 = h_3\mathrm{d}u_3$，可以得到关于哈密顿算符 ∇、$\nabla\phi$、$\nabla\cdot\boldsymbol{A}$、$\nabla\times\boldsymbol{A}$、$\nabla^2$ 及的一般表达式 Poisson 方程和 Laplace 方程的一般表达式。在这些表达式中 h_1、h_2、h_3 取不同的值，可以演化得到在相应坐标系下的对应运算式。

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