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The Fourier Expansion of Landau-Lifshitz Equations

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Abstract

Landau-Lifshitz equations describes magnetic phenomena of magnetization dynamics equation; studying the material is very meaningful. Scholars have constructed some explicit solutions. In this paper, we study the Landau-Lifshitz equations, which have no external magnetic field and the dissipation term is zero, by Fourier expansion. And we also give the iterative relations of finite terms of the equation.

Keywords

Landau-Lifshitz Equations, Fourier Expansion, Integral, Iteration

Landau-Lifshitz方程的Fourier展开

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摘　要

Landau-Lifshitz方程是描述磁性物质动态磁化现象的方程，而研究磁性物质是非常有意义的。目前学者们构造了部分精确解，本文利用Fourier展开研究无外加磁场且耗散项为零时的Landau-Lifshitz方程，给出方程有限项的迭代关系。

关键词

Landau-Lifshitz方程，Fourier展开，积分，迭代格式

1. 引言

1935 年，Landau 和 Lifshitz 在研究铁磁体的磁导率时，给出了如下的磁化运动方程[1] [2]：

$$\frac{\partial u}{\partial t}=\lambda_1 u\times H^e-\lambda_2 u\times\left(u\times H^e\right)$$

其中 λ_1， λ_2 是常数， $\lambda_2>0$ 效应场 H^e 被定义为

$$H^e=-\frac{\partial}{\partial t}E_{\mathrm{mag}}\left(u\right)=H-\nabla\left(\phi\left(u\right)\right)+\sum_{i,j=1}^{3}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}\text{。}$$

在静态条件下或阻尼很强时，得

$$\lambda_1 u\times H^e-\lambda_2 u\times\left(u\times H^e\right)=0$$

在没有外加磁场的情况下，方程可写为

$$\frac{\partial u}{\partial t}=\lambda_1 u\times\Delta u-\lambda_2 u\times\left(u\times\Delta u\right)$$

方程中的第一项表示磁化强度 u 绕效应场 Δu 的运动，第二项表示耗散，称之为耗散项或 Gilbert 项。

上述三个方程是人们对铁磁体的磁化运动研究从定性分析到定量分析的飞跃。弄清这三个模型对于进一步认识铁磁体的磁化运动规律很重要，因此它们引起了许多物理学家和数学家的重视。

2. 对 Landau-Lifshitz 方程的常微分求解

这里针对无外加磁场的 Landau-Lifshitz 方程

$$u_t=u\times\Delta u$$

本文优先考虑将方程化为常微分方程并求解，假设

$$u=v\left(a_0t+a_1x_1+a_2x_2+a_3x_3\right)=v\left(y\right)$$

通过 u 计算能够计算出

$$u_t=a_0v_y$$

$$\Delta u = u_{xx} = \left(a_1^2 + a_2^2 + a_3^2\right) v_{yy}$$

则

$$u \times \Delta u = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ \Delta u_1 & \Delta u_2 & \Delta u_3 \end{vmatrix} = \begin{cases} u_2 \cdot \Delta u_3 - u_3 \cdot \Delta u_2 \\ u_3 \cdot \Delta u_1 - u_1 \cdot \Delta u_3 \\ u_1 \cdot \Delta u_2 - u_2 \cdot \Delta u_1 \end{cases}$$

将 u_t， Δu 带入原方程便得到如下常微分方程组：

$$\begin{cases} a_0 v_{1y} = \lambda\left(v_2 \cdot v_{3yy} - v_3 \cdot v_{2yy}\right) \\ a_0 v_{2y} = \lambda\left(v_3 \cdot v_{1yy} - v_1 \cdot v_{3yy}\right) \\ a_0 v_{3y} = \lambda\left(v_1 \cdot v_{2yy} - v_2 \cdot v_{1yy}\right) \end{cases}$$

其中 $\lambda = a_1^2 + a_2^2 + a_3^2$。

对上述方程组等号两边同时对 y 求积分得：

$$\begin{cases} a_0 v_1 = \lambda\left(\int v_2 \cdot v_{3yy} \mathrm{d}y - \int v_3 \cdot v_{2yy} \mathrm{d}y\right) \\ a_0 v_2 = \lambda\left(\int v_3 \cdot v_{1yy} \mathrm{d}y - \int v_1 \cdot v_{3yy} \mathrm{d}y\right) \\ a_0 v_3 = \lambda\left(\int v_1 \cdot v_{2yy} \mathrm{d}y - \int v_2 \cdot v_{1yy} \mathrm{d}y\right) \end{cases}$$

分部积分得

$$\begin{aligned} \int v_2 \cdot v_{3yy} \mathrm{d}y &= \int v_2 \mathrm{d}v_{3y} \\ &= v_2 v_{3y} - \int v_{3y} \mathrm{d}v_2 \\ &= v_2 v_{3y} - \int v_{2y} \mathrm{d}v_3 \\ &= v_2 v_{3y} - v_{2y} v_3 + \int v_3 \cdot v_{2yy} \mathrm{d}y \end{aligned}$$

同理可得

$$\begin{cases} v_1 = \dfrac{\lambda}{a_0}\left(v_2 \cdot v_{3y} - v_3 \cdot v_{2y}\right) \\ v_2 = \dfrac{\lambda}{a_0}\left(v_3 \cdot v_{1y} - v_1 \cdot v_{3y}\right) \\ v_3 = \dfrac{\lambda}{a_0}\left(v_1 \cdot v_{2y} - v_2 \cdot v_{1y}\right) \end{cases}$$

3. Landau-Lifshitz 方程的 Fourier 展开

3.1. 通过计算 u_t， Δu 得

$$u_t = \sum_k^{\infty} \left[\begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix}_t \cos kx + \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix}_t \sin kx \right]$$

$$\Delta u = -\sum_{k}^{\infty} \lambda k^2 \left[\begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \cos kx + \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \sin kx \right]$$

将上述结果代入原方程

$$\sum_{k}^{\infty} \left[\begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix}_t \cos kx + \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix}_t \sin kx \right] = -\frac{1}{2} \lambda \sum_{k=0}^{\infty} \cdot \sum_{m=0}^{\infty} m^2 \left[\begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} \cos kx \cdot \cos mx + \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} \sin kx \cdot \cos mx + \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \cos kx \cdot \sin mx + \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \sin kx \cdot \sin mx \right]$$

运用积化和差公式化简得：

$$\sum_{k}^{\infty} \left[\begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix}_t \cos kx + \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix}_t \sin kx \right] = -\frac{1}{2} \lambda \sum_{k=0}^{\infty} \cdot \sum_{m=0}^{\infty} m^2 \left[\begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} \left(\cos(k+m)x + \cos(k-m)x \right) + \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} \left(\sin(k+m)x + \sin(k-m)x \right) + \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \left(\sin(k+m)x - \sin(k-m)x \right) - \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \left(\cos(k+m)x - \cos(k-m)x \right) \right]$$

3.2. 两边与 cosjx 做点乘后，在$[-\pi, \pi]$上分别积分

注意到

$$\int_{-\pi}^{\pi} \cos^2 jx \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos jx + 1}{2} \mathrm{d}x = \pi$$

则有

$$\pi\begin{pmatrix}a_{1j}\\a_{2j}\\a_{3j}\end{pmatrix}_t=-\frac{1}{2}\lambda\left[\sum_{k+m=j}\pi m^2\begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\end{pmatrix}\times\begin{pmatrix}a_{1m}\\a_{2m}\\a_{3m}\end{pmatrix}+\sum_{k-m=j}\pi m^2\begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\end{pmatrix}\times\begin{pmatrix}a_{1m}\\a_{2m}\\a_{3m}\end{pmatrix}\right]$$
$$+\frac{1}{2}\lambda\left[\sum_{k+m=j}\pi m^2\begin{pmatrix}b_{1k}\\b_{2k}\\b_{3k}\end{pmatrix}\times\begin{pmatrix}b_{1m}\\b_{2m}\\b_{3m}\end{pmatrix}-\sum_{k-m=j}\pi m^2\begin{pmatrix}b_{1k}\\b_{2k}\\b_{3k}\end{pmatrix}\times\begin{pmatrix}b_{1m}\\b_{2m}\\b_{3m}\end{pmatrix}\right]$$

合并同类项得

$$\begin{pmatrix}a_{1j}\\a_{2j}\\a_{3j}\end{pmatrix}_t=-\frac{1}{2}\lambda\left[\left(\sum_{k+m=j}+\sum_{k-m=j}\right)m^2\begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\end{pmatrix}\times\begin{pmatrix}a_{1m}\\a_{2m}\\a_{3m}\end{pmatrix}\right.$$
$$\left.-\left(\sum_{k+m=j}-\sum_{k-m=j}\right)m^2\begin{pmatrix}b_{1k}\\b_{2k}\\b_{3k}\end{pmatrix}\times\begin{pmatrix}b_{1m}\\b_{2m}\\b_{3m}\end{pmatrix}\right]$$

当 $j=0$ 时，有 $k=m=0$，此时

$$\begin{pmatrix}a_{10}\\a_{20}\\a_{30}\end{pmatrix}_t=-\frac{1}{2}\lambda\cdot 0$$

因此得出 $\begin{pmatrix}a_{10}\\a_{20}\\a_{30}\end{pmatrix}_t$ 为任意的常向量。

3.3. 两边与 $\sin jx$ 做点乘后，在 $[-\pi,\pi]$ 上分别积分

注意到

$$\int_{-\pi}^{\pi}\cos kx\sin jx\mathrm{d}x=0\qquad k\neq j;$$
$$\int_{-\pi}^{\pi}\sin kx\cdot\sin jx\mathrm{d}x=0\qquad k\neq j;$$
$$\int_{-\pi}^{\pi}\sin^2 jx\mathrm{d}x=\int_{-\pi}^{\pi}\frac{1-\cos jx}{2}\mathrm{d}x=\pi.$$

则有

$$\pi\begin{pmatrix}b_{1j}\\b_{2j}\\b_{3j}\end{pmatrix}_t=-\frac{1}{2}\lambda\left[\sum_{k+m=j}\pi m^2\begin{pmatrix}b_{1k}\\b_{2k}\\b_{3k}\end{pmatrix}\times\begin{pmatrix}a_{1m}\\a_{2m}\\a_{3m}\end{pmatrix}+\sum_{k-m=j}\pi m^2\begin{pmatrix}b_{1k}\\b_{2k}\\b_{3k}\end{pmatrix}\times\begin{pmatrix}a_{1m}\\a_{2m}\\a_{3m}\end{pmatrix}\right]$$
$$-\frac{1}{2}\lambda\left[\sum_{k+m=j}\pi m^2\begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\end{pmatrix}\times\begin{pmatrix}b_{1m}\\b_{2m}\\b_{3m}\end{pmatrix}-\sum_{k-m=j}\pi m^2\begin{pmatrix}a_{1k}\\a_{2k}\\a_{3k}\end{pmatrix}\times\begin{pmatrix}b_{1m}\\b_{2m}\\b_{3m}\end{pmatrix}\right]$$

得

$$\begin{pmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix}_t = -\frac{1}{2}\lambda\left[\left(\sum_{k+m=j} + \sum_{k-m=j}\right) m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} + \left(\sum_{k+m=j} - \sum_{k-m=j}\right) m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix}\right]$$

3.4. 有限项的 Landau-Lifshitz 方程

假设方程是有限项的，且前 $n-1$ 项都已知，用前 $n-1$ 项表示出第 n 项。因此假设 $k \le n, m \le n, j = n$ 有 [3] [4]

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix}_t = -\frac{1}{2}\lambda\left[\left(\sum_{k+m=j} + \sum_{k-m=j}\right) m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} - \left(\sum_{k+m=j} - \sum_{k-m=j}\right) m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix}\right]$$

$$\begin{pmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix}_t = -\frac{1}{2}\lambda\left[\left(\sum_{k+m=j} + \sum_{k-m=j}\right) m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} + \left(\sum_{k+m=j} - \sum_{k-m=j}\right) m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix}\right]$$

因为 $k-m=n, k \le n$ 且 $m \le n$，所以有如下结论成立：

$$\sum_{k-m=n} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} = 0$$

$$\sum_{k-m=n} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} = 0$$

故方程可以化简为

$$\begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix}_t = -\frac{1}{2}\lambda\left[\sum_{k+m=n} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} - \sum_{k+m=n} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix}\right]$$

同理

$$\sum_{k-m=n} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} = 0$$

$$\sum_{k-m=n} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} = 0$$

$$\begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix}_t = -\frac{1}{2}\lambda \left[\sum_{k+m=n} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} + \sum_{k+m=n} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \right]$$

因为 $k+m=n, k \le n$ 且 $m \le n$，所以有如下结论成立：

$$\begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix}_t = -\frac{1}{2}\lambda \left[n^2 \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix} \times \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix} + \sum_{k+m\le n-1} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} \right.$$
$$\left. - \sum_{k+m=n} n^2 \begin{pmatrix} b_{10} \\ b_{20} \\ b_{30} \end{pmatrix} \times \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix} + \sum_{k+m\le n-1} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \right]$$

$$\begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix}_t = -\frac{1}{2}\lambda \left[n^2 \begin{pmatrix} b_{10} \\ b_{20} \\ b_{30} \end{pmatrix} \times \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix} + \sum_{k+m\le n-1} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \right.$$
$$\left. + \sum_{k+m=n} n^2 \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix} \times \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix} + \sum_{k+m\le n-1} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} \right]$$

对于上面两个方程，令

$$\begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix} = x_n \text{，} \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix} = y_n$$

$$\sum_{k+m\le n-1} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{pmatrix} - \sum_{k+m\le n-1} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} = F_{n-1}$$

$$\sum_{k+m\le n-1} m^2 \begin{pmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} + \sum_{k+m\le n-1} m^2 \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} \times \begin{pmatrix} b_{1m} \\ b_{2m} \\ b_{3m} \end{pmatrix} = G_{n-1}$$

又

$$\begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix} \times \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix} = \begin{pmatrix} i & j & k \\ a_{10} & a_{20} & a_{30} \\ a_{1n} & a_{2n} & a_{3n} \end{pmatrix} = \begin{bmatrix} 0 & -a_{30} & a_{20} \\ a_{30} & 0 & -a_{10} \\ -a_{20} & a_{10} & 0 \end{bmatrix} \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix}$$

$$\begin{pmatrix} b_{10} \\ b_{20} \\ b_{30} \end{pmatrix} \times \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix} = \begin{pmatrix} i & j & k \\ b_{10} & b_{20} & b_{30} \\ a_{1n} & a_{2n} & a_{3n} \end{pmatrix} = \begin{bmatrix} 0 & -b_{30} & b_{20} \\ b_{30} & 0 & -b_{10} \\ -b_{20} & b_{10} & 0 \end{bmatrix} \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{pmatrix}$$

$$\begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix} \times \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix} = \begin{pmatrix} i & j & k \\ a_{10} & a_{20} & a_{30} \\ b_{1n} & b_{2n} & b_{3n} \end{pmatrix} = \begin{bmatrix} 0 & -a_{30} & a_{20} \\ a_{30} & 0 & -a_{10} \\ -a_{20} & a_{10} & 0 \end{bmatrix} \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix}$$

$$\begin{pmatrix} b_{10} \\ b_{20} \\ b_{30} \end{pmatrix} \times \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix} = \begin{pmatrix} i & j & k \\ b_{10} & b_{20} & b_{30} \\ b_{1n} & b_{2n} & b_{3n} \end{pmatrix} = \begin{bmatrix} 0 & -b_{30} & b_{20} \\ b_{30} & 0 & -b_{10} \\ -b_{20} & b_{10} & 0 \end{bmatrix} \begin{pmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{pmatrix}$$

为方便我们令

$$\begin{bmatrix} 0 & -a_{30} & a_{20} \\ a_{30} & 0 & -a_{10} \\ -a_{20} & a_{10} & 0 \end{bmatrix} = \mathrm{A}$$

$$\begin{bmatrix} 0 & -b_{30} & b_{20} \\ b_{30} & 0 & -b_{10} \\ -b_{20} & b_{10} & 0 \end{bmatrix} = \mathrm{B}$$

所以原方程可以写成如下形式

$$(x_n)_t = -\frac{1}{2}\lambda\left(n^2 \mathrm{A} x_n - n^2 \mathrm{B} y_n + F_{n-1}\right)$$

$$(y_n)_t = -\frac{1}{2}\lambda\left(n^2 \mathrm{B} x_n + n^2 \mathrm{A} y_n + G_{n-1}\right)$$

即

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix}_t = -\frac{1}{2}\lambda\left[\begin{pmatrix} n^2\mathrm{A} & -n^2\mathrm{B} \\ n^2\mathrm{B} & n^2\mathrm{A} \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} F_{n-1} \\ G_{n-1} \end{pmatrix}\right]$$

令

$$\begin{pmatrix} n^2\mathrm{A} & -n^2\mathrm{B} \\ n^2\mathrm{B} & n^2\mathrm{A} \end{pmatrix} = \mathrm{M}$$

两边同乘 $\mathrm{e}^{\int \frac{1}{2}\lambda \mathrm{M}\,\mathrm{d}t}$ 得[5] [6]

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -\frac{1}{2}\mathrm{e}^{\int \frac{1}{2}\lambda \mathrm{M}\mathrm{d}t}\int\left[\lambda \mathrm{e}^{\int \frac{1}{2}\lambda \mathrm{M}\mathrm{d}t}\begin{pmatrix} F_{n-1} \\ G_{n-1} \end{pmatrix}\right]\mathrm{d}t 。$$

参考文献 (References)

[1] 杨干山. 多维 Landau-Lifshitz 方程的型黏性解、Blow up 解和整体解[D]: [博士学位论文]. 北京: 中国工程物理研究院, 2002: 1-11.

[2] 郭柏灵, 丁时进. 自旋波与铁磁链方程[M]. 浙江: 浙江科学技术出版社, 2002: 1-12.

[3] Guo, B. and Yang, G.S. (2001) Some Exact Nontrivial Global Solutions with Values in Unit Sphere for Two-Dimensional Landau-Lifshitz Equations. *Journal of Mathematical Physics*, **42**, 5223-5227. https://doi.org/10.1063/1.1402955

[4] Guo, B., Han, Y.Q. and Yang, G. S. (2001) Exact Blow-Up Solutions for Multidimensional Landau-Lifshitz Equations. *Advances in mathematics* (*China*), **30**, 91-93.

[5] 刘利敏. Landau-Lifshitz 方程解的一些研究[D]: [博士学位论文]. 重庆: 重庆大学, 2010.

[6] Vretblad, A. Fourier Analysis and Its Applications. 北京: 北京科学技术出版社, 2002: 1-12.